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Baumslag-Solitar groups and their von Neumann algebras

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Dissertation presented in partial fulfillment of the requirements for the degree of Doctor of Science

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Abstract

We examine both the group von Neumann algebras of the Baumslag-Solitar groups and the crossed product von Neumann algebras of some of their actions. In the case of the group von Neumann algebras, we show that the rational number $|n/m|$ is an invariant of $L(\mathrm{BS}(n,m))$. Concretely, if $L(\mathrm{BS}(n,m))$ and $L(\mathrm{BS}(p,q))$ are isomorphic and nonamenable, then $|n/m| = |p/q|^{\pm 1}$. In the case of the crossed products, we show that $\mathrm{BS}(n,m)$ is an invariant of $L^\infty(X) \rtimes \mathrm{BS}(n,m)$ whenever the canonical almost normal abelian subgroup acts aperiodically. More precisely, let $\mathrm{BS}(n,m) \curvearrowright X$ and $\mathrm{BS}(p,q) \curvearrowright Y$ be two ergodic essentially free probability measure preserving actions of nonamenable Baumslag-Solitar groups whose canonical almost normal abelian subgroups act aperiodically, then an isomorphism between the crossed products of the actions forces $\mathrm{BS}(n,m) \cong \mathrm{BS}(p,q)$ when $|n| \neq |m|$ and $\mathrm{BS}(n,m) \cong \mathrm{BS}(p,\pm q)$ when $|n| = |m|$.

Beknopte samenvatting

We onderzoeken zowel de groeps-von Neumannalgebra's van de Baumslag-Solitar groepen als de gekruiste product von Neumannalgebra's van enkele van hun acties. In het geval van de groeps-von Neumannalgebra's tonen we aan dat het rationaal getal $|n/m|$ een invariant is van $L(\mathrm{BS}(n,m))$. Concreter, als $L(\mathrm{BS}(n,m))$ en $L(\mathrm{BS}(p,q))$ isomorf en niet-amenabel zijn, dan is $|n/m| = |p/q|^{\pm 1}$. In het geval van de gekruiste producten tonen we aan dat $\mathrm{BS}(n,m)$ een invariant is van $L^\infty(X) \rtimes \mathrm{BS}(n,m)$ wanneer de kanonieke bijna-normale, abelse deelgroep aperiodisch werkt. Preciezer, stel dat $\mathrm{BS}(n,m) \curvearrowright X$ en $\mathrm{BS}(p,q) \curvearrowright Y$ twee ergodische, essentieel vrije, kansmaat bewarende acties van niet-amenabele Baumslag-Solitar groepen zijn wier kanonieke bijna-normale, abelse deelgroepen aperiodisch werken, dan impliceert het bestaan van een isomorfisme tussen de gekruiste producten van de acties dat $\mathrm{BS}(n,m) \cong \mathrm{BS}(p,q)$ als $|n| \neq |m|$ en $\mathrm{BS}(n,m) \cong \mathrm{BS}(p,\pm q)$ als $|n| = |m|$.

Contents

Introduction

A *von Neumann algebra* is an algebra of bounded linear operators on a Hilbert space $\mathcal{H}$ that is self-adjoint, weak operator closed and contains the identity operator $1_{\mathcal{H}}$. For a subset M of $\mathrm{B}(\mathcal{H})$, we denote by M' its commutant in $\mathrm{B}(\mathcal{H})$:

$$M' = \{S \in \mathrm{B}(\mathcal{H}) \mid ST = TS,\ T \in M\}.$$

The commutant M'' of M' is then called the *bicommutant* of M. By the famous bicommutant theorem of von Neumann ([vN29]) one can equivalently define a von Neumann algebra as a self-adjoint subalgebra of $\mathrm{B}(\mathcal{H})$ that is equal to its bicommutant.

It is clear from the definition that both $\mathrm{B}(\mathcal{H})$ and $\mathbb{C}1_{\mathcal{H}}$ are von Neumann algebras. Of course, these examples are not very interesting. Instead, one can for instance look at the class of group von Neumann algebras and crossed product von Neumann algebras.

A *group von Neumann algebra* $L(\Gamma)$ is a von Neumann algebra that we associate to a countable discrete group Γ. It is nothing else than the von Neumann algebra generated by the image of the left regular representation. Although the definition of a group von Neumann algebra is fairly simple, their classification is most definitely not. Some of the deepest open problems in functional analysis center around the classification of group von Neumann algebras $L(\Gamma)$ associated with certain natural families of countable groups Γ. In the case of the free groups, this becomes the famous free group factor problem asking whether $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ when $n, m \geq 2$ and $n \neq m$. By the work of Dykema [Dy94] and Rădulescu [Ra94] it is known that either they are all pairwise isomorphic or all pairwise nonisomorphic.

We say that a group Γ is icc if the conjugacy class of every nontrivial element is infinite. It turns out that Γ is icc if and only if $L(\Gamma)$ is a factor, i.e. its center is trivial. A group Γ has property (T) if every unitary representation of Γ on a Hilbert space having almost invariant vectors has a nonzero invariant vector. For icc property (T) groups we have Connes's rigidity conjecture ([Co80]) asserting that an isomorphism $L(\Gamma) \cong L(\Lambda)$ between the von Neumann algebras entails an isomorphism $\Gamma \cong \Lambda$ between the icc property (T) groups.

As a consequence of Connes's uniqueness theorem of injective II_1 factors ([Co75]), the group von Neumann algebras of all amenable (i.e. groups admitting a left-invariant mean) icc groups are isomorphic to each other. In the nonamenable case, many nonisomorphic groups Γ are known to have nonisomorphic group von Neumann algebras $L(\Gamma)$. Nevertheless, concerning the classification of group von Neumann algebras of natural families of groups, e.g. lattices in simple Lie groups, little is known. A notable exception however is [CH88] where it is shown that for $n \neq m$, lattices in $\mathrm{Sp}(n, 1)$, respectively $\mathrm{Sp}(m, 1)$, have nonisomorphic group von Neumann algebras.

Since 2001, Popa has been developing a new arsenal of techniques, called deformation/rigidity theory. This theory has provided several classes $\mathcal{G}$ of groups such that an isomorphism $L(\Gamma) \cong L(\Lambda)$ with both $\Gamma, \Lambda \in \mathcal{G}$ entails the isomorphism $\Gamma \cong \Lambda$. By [Po04], this holds in particular when $\mathcal{G}$ is the class of wreath product groups of the form $(\mathbb{Z}/2\mathbb{Z}) \wr G$ with G an icc property (T) group.

In [IPV10], the first W^*-superrigidity theorems for group von Neumann algebras were discovered, yielding icc groups Γ such that an isomorphism $L(\Gamma) \cong L(\Lambda)$ with Λ an *arbitrary* countable group, implies that $\Gamma \cong \Lambda$. The groups Γ discovered in [IPV10] are generalized wreath products of a special form. In [BV12], it was then shown that one can actually take $\Gamma = (\mathbb{Z}/2\mathbb{Z})^{(G)} \rtimes (G \times G)$ with G ranging over a large family of nonamenable groups including the free groups $\mathbb{F}_n$, $n \geq 2$.

A *crossed product von Neumann algebra* $L^\infty(X) \rtimes \Gamma$ is a von Neumann algebra that we associate to a group action $\Gamma \curvearrowright X$ of a countable discrete group Γ on a standard probability space X. It is defined as the von Neumann algebra generated by $L^\infty(X)$ and the unitaries $(u_g)_{g \in \Gamma}$ satisfying $u_g^* f u_g = f(g\ \cdot)$ and $u_g u_h = u_{gh}$, for every $f \in L^\infty(X)$ and $g, h \in \Gamma$.

We say that a group action $\Gamma \curvearrowright X$ is essentially free if the Borel set $\{x \mid g{\cdot}x = x\}$ has measure zero whenever $g \neq e$. A group action is called ergodic if every Γ-invariant Borel subset has measure 0 or 1. It turns out that the crossed product $L^\infty(X) \rtimes \Gamma$ of an essentially free ergodic probability measure preserving (pmp) action $\Gamma \curvearrowright X$ is a factor. In that case, we have that $L^\infty(X)$ is a Cartan subalgebra of $L^\infty(X) \rtimes \Gamma$, meaning that $L^\infty(X)$ is maximal abelian and the set of all unitaries in $L^\infty(X) \rtimes \Gamma$ normalizing $L^\infty(X)$ generates the whole crossed product as a von Neumann algebra.

Cartan subalgebras play a crucial role in the classification of crossed product von Neumann algebras. Indeed, by Singer's theorem ([Si55]), if $L^\infty(X) \rtimes \Gamma$ has a unique Cartan subalgebra up to unitary conjugacy, then $L^\infty(X) \rtimes \Gamma$ completely 'remembers' the orbit equivalence relation on X given by $x \sim y$ if and only if $x \in \Gamma \cdot y$.

In [PV11], Popa and Vaes showed that the nonabelian free groups $\mathbb{F}_n$ are so-called $\mathcal{C}$*-rigid*, meaning that whenever $\mathbb{F}_n \curvearrowright X$ is an essentially free ergodic pmp action, the crossed product $L^\infty(X) \rtimes \mathbb{F}_n$ has a unique Cartan subalgebra up to unitary conjugacy. So an isomorphism $L^\infty(X) \rtimes \mathbb{F}_n \cong L^\infty(Y) \rtimes \mathbb{F}_m$, arising from arbitrary essentially free ergodic pmp actions $\mathbb{F}_n \curvearrowright X$, $\mathbb{F}_m \curvearrowright Y$ of free groups forces the actions to be orbit equivalent. From earlier work of Gaboriau ([Ga99],[Ga01]), this in turn forces $n = m$.

It should be pointed out that we can not always expect the crossed product $L^\infty(X) \rtimes \Gamma$ to 'remember' much from the action $\Gamma \curvearrowright X$. Indeed, Connes's uniqueness theorem of injective II_1 factors implies that the crossed products of all essentially free ergodic pmp actions of all infinite amenable groups are isomorphic to each other.

In this thesis we examine both the group von Neumann algebras of the Baumslag-Solitar groups and the crossed product von Neumann algebras of some of their actions. Recall that for every pair (n, m) of nonzero integers, the *Baumslag-Solitar group* $\mathrm{BS}(n, m)$ is defined as the group generated by a and b subject to the relation $ba^n b^{-1} = a^m$. So,

$$\mathrm{BS}(n, m) := \langle a, b \mid ba^n b^{-1} = a^m \rangle.$$

The Baumslag-Solitar groups were introduced in [BS62] as the first examples of two generator non-Hopfian groups with a single defining relation. Ever since they have been used as examples and counterexamples for numerous group theoretic phenomena. It is therefore a natural problem to classify the group von Neumann algebras and crossed product von Neumann algebras arising from these groups.

Whenever $2 \leq |n|, |m|$, $\mathrm{BS}(n, m)$ contains a copy of the free group $\mathbb{F}_2$ and hence is nonamenable. In every other case, the group is solvable and therefore amenable. From [Mo91], we know that $\mathrm{BS}(n, m) \cong \mathrm{BS}(p, q)$ if and only if $\{n, m\} = \{\varepsilon p, \varepsilon q\}$ for some $\varepsilon \in \{1, -1\}$. By the work of Farb and Mosher in [FM98] and Whyte in [Wh01], the Baumslag-Solitar groups are also classified up to quasi-isometry. Recall that two metric spaces (M_1, d_1) and (M_2, d_2) are quasi-isometric if there exists a map $f : M_1 \to M_2$ and constants $A \geq 1$, $B \geq 0$ and $C \geq 0$ such that the C-neighbourhood of $f(M_1)$ is the whole of M_2 and $A^{-1} d_1(x, y) - B \leq d_2(f(x), f(y)) \leq A\, d_1(x, y) + B$ for all $x, y \in M_1$. Two finitely generated groups Γ_1 and Γ_2 are then called quasi-isometric, if they are quasi-isometric when equipped with the word-metric. Farb and Mosher showed that given integers $n, m > 0$, the groups $\mathrm{BS}(1, n)$ and $\mathrm{BS}(1, m)$ are quasi-isometric if and only if there exist integers $r, j, k > 0$ such that $n = r^j$ and $m = r^k$. On the other hand, Whyte showed that all groups $\mathrm{BS}(n, m)$ with $1 < n < m$ are quasi-isometric to each other.

As was mentioned above, we examine in this thesis the group von Neumann algebras of the Baumslag-Solitar groups as well as the crossed product von Neumann algebras of some of their actions. Let us explain the structure of the thesis and give an overview of our main results.

In Chapter 1 we provide all preliminaries needed to understand the main results and their proofs.

In Chapter 2 we prove our first main result. It is a partial classification of the Baumslag-Solitar group von Neumann algebras which was published in [MV13]. We show that the rational number $|n/m|$ is an invariant of the von Neumann algebra. Concretely, if $L(\mathrm{BS}(n, m))$ and $L(\mathrm{BS}(p, q))$ are isomorphic and nonamenable, then $|n/m| = |p/q|^{\pm 1}$. On the other hand, by Connes's uniqueness theorem of injective II_1 factors, we know that $L(\mathrm{BS}(1, n))$ and $L(\mathrm{BS}(1, m))$ are isomorphic, whenever $|n|, |m| > 1$. It is interesting to note that these classification results are in sheer contrast with the classification up to quasi-isometry discussed above. Indeed, we have seen that in the nonamenable case there is no distinction possible up to quasi-isometry between the groups, while in the amenable case there is. We are not aware of another family of groups admitting this contrasting behaviour.

Chapter 3 deals with the second main result. It is a rigidity result for crossed products arising from certain actions of Baumslag-Solitar groups which goes as follows. For $i \in \{1, 2\}$, let $\mathrm{BS}(n_i, m_i) \curvearrowright (X_i, \mu_i)$ be an essentially free ergodic pmp action of a nonamenable Baumslag-Solitar group such that $\langle a_i^k \rangle \curvearrowright X_i$ is ergodic for every nonzero positive integer k. If the crossed products $L^\infty(X_1) \rtimes \mathrm{BS}(n_1, m_1)$ and $L^\infty(X_2) \rtimes \mathrm{BS}(n_2, m_2)$ are isomorphic, then

- $\mathrm{BS}(n_1, m_1) \cong \mathrm{BS}(n_2, m_2)$, if $|n_1| \neq |m_1|$;
- $\mathrm{BS}(n_1, m_1) \cong \mathrm{BS}(n_2, \pm m_2)$, if $|n_1| = |m_1|$.

We want to point out to the reader that we do not show a uniqueness result for Cartan subalgebras to prove this, but instead use a purely von Neumann algebraic approach.

Finally, we end with Chapter 4 where we present our conclusion and give some future perspectives.

Chapter 1

Preliminaries

1.1 Von Neumann algebras

Let $\mathcal{H}$ be a complex Hilbert space with inner-product $\langle\cdot,\cdot\rangle$, and let $\mathrm{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. We can consider the two following topologies on $\mathrm{B}(\mathcal{H})$:

- the *strong operator topology*, that is, the topology on $\mathrm{B}(\mathcal{H})$ generated by the seminorms
$$p_\xi(x) = ||x\xi||, \quad \xi \in \mathcal{H};$$
- the *weak operator topology*, that is, the topology on $\mathrm{B}(\mathcal{H})$ generated by the seminorms
$$p_{\xi,\eta}(x) = |\langle x\xi, \eta\rangle|, \quad \xi, \eta \in \mathcal{H}.$$

In 1936, Murray and von Neumann introduced the notion of what is now called a *von Neumann algebra* ([MvN36]). They defined it as being an algebra of bounded linear operators on a Hilbert space $\mathcal{H}$ that is

- *self-adjoint*, i.e. closed under taking adjoints;
- weak operator closed (equivalently, strong operator closed);
- containing the *identity operator* $1_{\mathcal{H}}$.

Next to this analytic definition of a von Neumann algebra, there also exists an algebraic definition. This is exactly the content of Theorem 1.1.1 below.

For a subset M of $\mathrm{B}(\mathcal{H})$, we denote by M' its commutant in $\mathrm{B}(\mathcal{H})$:

$$M' := \{S \in \mathrm{B}(\mathcal{H}) \mid ST = TS,\ T \in M\}.$$

The commutant M'' of M' is then called the *bicommutant* of M.

Theorem 1.1.1 (Bicommutant theorem, [vN29])**.** *Let $M \subset \mathrm{B}(\mathcal{H})$ be a self-adjoint subalgebra such that $1_{\mathcal{H}} \in M$. Then the following conditions are equivalent:*

1. *M is weak operator closed;*
2. *M is strong operator closed;*
3. *$M = M''$.*

So we see that a von Neumann algebra is equivalently also a self-adjoint algebra of bounded linear operators on a Hilbert space that is equal to its own bicommutant.

Definition 1.1.2. Two von Neumann algebras M and N are called *isomorphic* if there exists a $*$-isomorphism $\alpha : M \to N$.

It is clear that both $\mathrm{B}(\mathcal{H})$ and $\mathbb{C}1_{\mathcal{H}}$ are von Neumann algebras. Of course, these examples are not very interesting. More interesting examples are for instance the *group von Neumann algebras* and the *crossed product von Neumann algebras*. The next two sections are devoted to these specific classes of von Neumann algebras.

Another possibility for constructing a von Neumann algebra is through the use of tensor products. For that let $M \subset \mathrm{B}(\mathcal{H})$ and $N \subset \mathrm{B}(\mathcal{K})$ be two von Neumann algebras. The algebraic tensor product $M \odot N$ of M and N acts on the Hilbert tensor product $\mathcal{H} \otimes \mathcal{K}$ as follows:

$$(x \otimes y)(\xi \otimes \eta) = (x\xi) \otimes (y\eta), \quad x \in M, y \in N, \xi \in \mathcal{H}, \eta \in \mathcal{K}.$$

The strong operator closure $M \overline{\otimes} N$ of $M \odot N$ is a von Neumann algebra called the *tensor product* of M and N.

Other methods for constructing von Neumann algebras will be introduced throughout the thesis whenever necessary.

1.2 Group von Neumann algebras

Throughout this thesis we always denote by Γ a countable discrete group.

So let Γ be a countable discrete group. The *left regular representation* $\lambda : \Gamma \to \mathrm{B}(l^2(\Gamma))$ is defined as

$$\lambda(g)\delta_h = \delta_{gh}, \quad g, h \in \Gamma,$$

where $(\delta_h)_{h\in\Gamma}$ denotes the canonical orthonormal basis of $l^2(\Gamma)$.

Definition 1.2.1 ([MvN36])**.** The von Neumann algebra generated by $\lambda(\Gamma)$ is called the *group von Neumann algebra* of Γ and is denoted by $L(\Gamma)$. Equivalently, the von Neumann algebra $L(\Gamma)$ is the weak operator closure of the linear span of $\{\lambda(g) \mid g \in \Gamma\}$.

Inside $L(\Gamma)$ we denote the unitary $\lambda(g)$ by u_g, for every $g \in \Gamma$.

The *center* of a von Neumann algebra M is denoted by $\mathcal{Z}(M)$ and is defined as $\mathcal{Z}(M) = M \cap M'$. We call M a *factor* if its center is trivial, i.e. when $\mathcal{Z}(M) = \mathbb{C}1$. In that case M is indecomposable in the sense that it cannot be written as the direct sum of two von Neumann algebras. In the case of group von Neumann algebras, it turns out that $L(\Gamma)$ is a factor if and only if Γ is *icc*, i.e. the conjugacy class of every nontrivial element of Γ is infinite ([MvN43, Lemma 5.3.4]). Some easy examples of icc groups are for instance the nonabelian free groups $\mathbb{F}_{n\geq 2}$ and the group S_∞ of all finite permutations on $\mathbb{N}$.

On another note, consider the map $\tau : L(\Gamma) \to \mathbb{C}$ defined by $\tau(x) = \langle x\delta_e, \delta_e\rangle$. Then τ is a linear functional satisfying $\tau(1) = 1$ and is

- *positive*, i.e. $\tau(x^*x) \geq 0$ for all $x \in L(\Gamma)$;
- *faithful*, i.e. $\tau(x^*x) = 0$ if and only if $x = 0$;
- *tracial*, i.e. $\tau(xy) = \tau(yx)$ for all $x, y \in L(\Gamma)$;
- *normal*, i.e. τ is continuous with respect to the weak operator topology when restricted to the unit ball.

A positive linear functional on a von Neumann algebra mapping the identity operator to 1 is called a *state*. A von Neumann algebra (M, τ) is called *tracial* if it comes equipped with a normal faithful tracial state τ. Hence $(L(\Gamma), \tau)$ is a tracial von Neumann algebra for every countable discrete group Γ.

Definition 1.2.2. A *II_1 factor* is an infinite dimensional factor admitting a normal faithful tracial state.

Following the previous definition, we have that $L(\Gamma)$ is a II_1 factor if and only if Γ is icc. In particular, $L(\mathbb{F}_{n\geq 2})$ and $L(S_\infty)$ are examples of II_1 factors.

1.3 Group actions and crossed product von Neumann algebras

Another method of constructing von Neumann algebras is the so called crossed product construction where we start with a group action and build a von Neumann algebra out of it. Let us first give some basics on group actions before we introduce the construction itself.

1.3.1 Group actions

The group actions that we consider are defined on what we call a standard probability space. Let us introduce these spaces and explain their connection with abelian von Neumann algebras.

Definition 1.3.1.

- A *Borel space* is a set endowed with a σ-algebra.
- A *standard Borel space* is a Borel space that is isomorphic to some separable complete metric space with the Borel σ-algebra.
- A *standard measure space* (X, μ) is a standard Borel space X equipped with a measure μ.
- A *standard probability space* (X, μ) is a standard measure space satisfying $\mu(X) = 1$.

A Borel isomorphism $\Delta : X \to Y$ between two standard probability spaces is called *nonsingular* whenever it is null-set preserving.

Definition 1.3.2. Let (X, μ) and (Y, η) be standard probability spaces.

- An *isomorphism* between X and Y is a nonsingular Borel isomorphism $\Delta : X' \to Y'$, where $X' \subset X$ and $Y' \subset Y$ are conegligible subsets;
- A *measure preserving isomorphism* between X and Y is a measure preserving Borel isomorphism $\Delta : X' \to Y'$, where $X' \subset X$ and $Y' \subset Y$ are conegligible subsets.

Two (measure preserving) isomorphisms that coincide on a conegligible subset of X will be identified.

As it turns out, the standard probability spaces can easily be classified.

Theorem 1.3.3 (Theorem 17.41, [Ke95]). *Let (X, μ) be a standard probability space, then there exists a measure preserving isomorphism from (X, μ) onto $[0, 1]$, where $[0, 1]$ is equipped with its natural Borel structure and a convex combination of the Lebesgue measure and a discrete probability measure.*

Now that we have introduced the basics of standard probability spaces, it is important to understand why we only consider this specific kind of probability spaces. It turns out that the class of standard probability spaces has some big advantages over the class of all probability spaces. Two of these advantages are given in Theorem 1.3.5 and Theorem 1.3.6. We start however with the following result.

Proposition 1.3.4 ([Di81, Theorem 2, page 132]). *Let (X, μ) be a standard probability space. Set $A = L^\infty(X, \mu)$.*

1. *For $f \in L^\infty(X, \mu)$, consider M_f to be the multiplication operator by f on $L^2(X, \mu)$. Then M_f is a bounded operator.*
2. *When considering A inside $\mathrm{B}(L^2(X, \mu))$, we have that $A = A'$. In particular A is a von Neumann algebra acting on a separable Hilbert space and A is abelian.*

The converse of the previous proposition also holds, meaning that every abelian von Neumann algebra acting on a separable Hilbert space is of the form $L^\infty(X, \mu)$, although it need not be represented on $L^2(X, \mu)$.

Theorem 1.3.5 ([Di81, Theorem 1, page 132]). *Any abelian von Neumann algebra A on a separable Hilbert space is isomorphic to $L^\infty(X, \mu)$ for some standard probability space (X, μ). Furthermore, if A comes equipped with a normal faithful tracial state τ, then the identification with $L^\infty(X, \mu)$ can be chosen to preserve the traces (where $L^\infty(X, \mu)$ comes equipped with the trace $\tau_\mu : L^\infty(X, \mu) \to \mathbb{C} : f \mapsto \int f d\mu$).*

This shows that there is a one-to-one correspondence between standard probability spaces and abelian von Neumann algebras acting on a separable Hilbert space. The next theorem says that also their isomorphisms are in a one-to-one correspondence.

Theorem 1.3.6 ([Bo07, Theorem 9.5.1]). *Let (X, μ) and (Y, η) be standard probability spaces. The set $\mathrm{Iso}(X, Y)$ of isomorphisms from X onto Y can be identified with the set $\mathrm{Iso}(L^\infty(X), L^\infty(Y))$ of isomorphisms from $L^\infty(X, \mu)$ onto $L^\infty(Y, \eta)$ through the map*

$$\alpha : \mathrm{Iso}(X, Y) \to \mathrm{Iso}(L^\infty(X), L^\infty(Y)) : \Delta \mapsto \Delta_*,$$

where $\Delta_*(f) = f \circ \Delta^{-1}$ *for all* $f \in L^\infty(X,\mu)$*. Furthermore, the map* α *also maps the measure preserving isomorphisms in* $\mathrm{Iso}(X,Y)$ *onto the trace-preserving isomorphisms in* $\mathrm{Iso}(L^\infty(X), L^\infty(Y))$ *(w.r.t.* τ_μ*, resp.* τ_η*).*

Now that we know the basics of standard probability spaces and understand their importance, we can finally turn our attention towards group actions. We begin with the definition.

Definition 1.3.7. A (*nonsingular*) *group action* $\Gamma \curvearrowright^\alpha (X,\mu)$ of a countable discrete group Γ on a standard probability space (X,μ) is a group homomorphism $\alpha : \Gamma \to \mathrm{Aut}(X,\mu)$, where $\mathrm{Aut}(X)$ is the group of automorphisms of X. We call the action $\Gamma \curvearrowright^\alpha (X,\mu)$ *probability measure preserving*, or *pmp*, if $\alpha(g)$ is measure preserving for every $g \in \Gamma$.

When $\Gamma \curvearrowright^\alpha (X,\mu)$ is a group action, we denote $\alpha(g)(x)$ by $g \cdot x$ for every $g \in \Gamma$ and $x \in X$.

An easy example of a group action can be constructed from the rotation of the circle over an angle $\theta \in [0, 2\pi[$. Concretely, we have that $\mathbb{Z} \curvearrowright \mathbb{T}$ with $z \cdot x = e^{iz\theta}x$ defines a group action of $\mathbb{Z}$ on the circle $\mathbb{T}$.

It is also important to be able to express when two group actions are 'equivalent'. There are many different ways of doing this. Here we introduce three of them.

Definition 1.3.8. Two group actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\eta)$ are called

- *conjugated*, if there exists an isomorphism $\delta : \Gamma \to \Lambda$ and an isomorphism $\Delta : X \to Y$ such that $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$ for every $g \in \Gamma$ and almost every $x \in X$;
- *orbit equivalent*, if there exists an isomorphism $\Delta : X \to Y$ such that $\Delta(\Gamma \cdot x) = \Lambda \cdot \Delta(x)$ for almost every $x \in X$;
- *stably orbit equivalent*, if there exist nonnegligible Borel subsets $X' \subset X$ and $Y' \subset Y$ that meet the orbit of almost every element and if there exists an isomorphism $\Delta : X' \to Y'$ such that $\Delta(\Gamma \cdot x \cap X') = \Lambda \cdot \Delta(x) \cap Y'$ for almost every $x \in X'$.

Most of the time, we will assume our group actions to have the following two properties . We will see in the next subsection that they will play an important role when we look at the crossed product associated to the group action.

Definition 1.3.9. A group action $\Gamma \curvearrowright (X,\mu)$ is called

- *ergodic*, if every Γ-invariant Borel subset of X has measure 0 or 1;

- *essentially free*, if for every $g \in \Gamma \setminus \{e\}$ the Borel set $\{x \in X \mid g \cdot x = x\}$ has measure 0.

When looking back at the action $\mathbb{Z} \curvearrowright^{\alpha} \mathbb{T}$ given by the rotation of the circle over an angle $\theta \in [0, 2\pi[$, we get that $\theta \notin \pi\mathbb{Q}$ is equivalent with both α being ergodic and α being essentially free.

Finally, note that every action $\Gamma \curvearrowright (X, \mu)$ induces an action $\Gamma \curvearrowright L^\infty(X, \mu)$, i.e. a group homomorphism from Γ to $\mathrm{Aut}(L^\infty(X, \mu))$, by

$$(g \cdot f)(x) = f(g^{-1} \cdot x).$$

In fact, Theorem 1.3.6 states that every action $\Gamma \curvearrowright L^\infty(X, \mu)$ is of this specific form.

1.3.2 Crossed product von Neumann algebras

As was mentioned at the end of the previous subsection, a group action on a standard probability space (X, μ) can be identified with a group action on the von Neumann algebra $L^\infty(X, \mu)$. Let us therefore introduce the crossed product construction for this more general setting of group actions on von Neumann algebras.

Definition 1.3.10 ([MvN36])**.** Let $\Gamma \curvearrowright^{\alpha} M$ be an action of a countable group Γ on a von Neumann algebra $M \subset \mathrm{B}(\mathcal{H})$. We define the *crossed product von Neumann algebra* $M \rtimes \Gamma \subset \mathrm{B}(\mathcal{H} \otimes l^2(\Gamma))$ as

$$M \rtimes \Gamma = \{au_g \mid a \in M, g \in \Gamma\}'',$$

where for $\xi \otimes \delta_h \in \mathcal{H} \otimes l^2(\Gamma)$ we have $a(\xi \otimes \delta_h) = \alpha_h(a)\xi \otimes \delta_h$ and $u_g(\xi \otimes \delta_h) = \xi \otimes \delta_{hg^{-1}}$.

Note that $u_g a u_g^* = \alpha_g(a)$ in $M \rtimes \Gamma$ for every $a \in M$ and $g \in \Gamma$.

The following properties of group actions on von Neumann algebras play an important role for the corresponding crossed product von Neumann algebra.

Definition 1.3.11. A group action $\Gamma \curvearrowright^{\alpha} M$ is called

- *ergodic*, if every Γ-invariant element of M is inside $\mathbb{C}1$;
- *properly outer*, if whenever $a \in M$ and $g \in \Gamma \setminus \{e\}$, $ax = \alpha_g(x)a$ for every $x \in M$ forces a to be zero.

Note that $\Gamma \curvearrowright L^\infty(X)$ is ergodic if and only if $\Gamma \curvearrowright X$ is ergodic, see e.g. Theorem 1.6 of [Wa82]. Also $\Gamma \curvearrowright L^\infty(X)$ is properly outer if and only if $\Gamma \curvearrowright X$ is essentially free, see e.g. the proof of [Ho13, Proposition 10].

Usually when considering a group action $\Gamma \curvearrowright M$ we assume that M comes equipped with a normal faithful tracial state and the action itself is trace preserving. In that case we can define a trace τ' on $M \rtimes \Gamma$ given by $\tau'(au_g) = \tau(a)\delta_{e,g}$. When $\Gamma \curvearrowright M$ is trace preserving, we have that $\Gamma \curvearrowright M$ is properly outer if and only if $M' \cap (M \rtimes \Gamma) = \mathcal{Z}(M)$, see e.g. [Tak03, Proposition XI.2.25]. So whenever $\Gamma \curvearrowright M$ is trace preserving, $M \rtimes \Gamma$ is a factor if $\Gamma \curvearrowright M$ is properly outer and $\Gamma \curvearrowright \mathcal{Z}(M)$ is ergodic. Altogether, we see that $M \rtimes \Gamma$ is a II_1 factor if $\Gamma \curvearrowright (M, \tau)$ is an ergodic properly outer trace preserving action of an infinite group. In particular $L^\infty(X) \rtimes \Gamma$ is a II_1 factor whenever $\Gamma \curvearrowright (X, \mu)$ is an essentially free ergodic pmp action of an infinite group Γ.

For any inclusion $A \subset M$ of von Neumann algebras, the *normalizer* ([Di54]) $N_M(A)$ is defined as

$$N_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}.$$

A von Neumann subalgebra A of M is called *regular* if $N_M(A)$ generates the whole of M as a von Neumann algebra.

Definition 1.3.12 ([FM77b])**.** Let M be a II_1 factor. A subalgebra $A \subset M$ is called a *Cartan subalgebra* if A is maximal abelian (i.e. $A = A' \cap M$) and regular.

Let $\Gamma \curvearrowright (X, \mu)$ be an essentially free ergodic pmp action of an infinite group Γ. We already mentioned that $M = L^\infty(X) \rtimes \Gamma$ is a II_1 factor. Set $A = L^\infty(X)$ and view A as a subalgebra of M. Then A is maximal abelian inside M, since the action is essentially free and pmp. Furthermore, for every $g \in \Gamma$, the element u_g belongs to $N_M(A)$. Since M is generated by A and $\{u_g \mid g \in \Gamma\}$, we find that A is also regular. Altogether, we see that A is a Cartan subalgebra of M. In particular, we call this a Cartan subalgebra of *group measure space type*.

The importance of Cartan subalgebras of group measure space type is illustrated by the following famous theorem of Singer.

Theorem 1.3.13 (Singer, [Si55])**.** *Let $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$ be essentially free ergodic pmp actions of countable groups. Assume that $\Delta : X \to Y$ is a measure preserving isomorphism. Denote by Δ_* the corresponding trace preserving isomorphism $\Delta_* : L^\infty(X) \to L^\infty(Y)$ given by $\Delta_*(f) = f \circ \Delta^{-1}$. The following two statements are equivalent.*

1. *Δ is an orbit equivalence;*

2. Δ_* *extends to a* $*$*-isomorphism* $L^\infty(X) \rtimes \Gamma \to L^\infty(Y) \rtimes \Lambda$*.*

From Singer's theorem, we can deduce the following important fact. Let $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$ be essentially free ergodic pmp actions of infinite countable groups. If the crossed products $L^\infty(X) \rtimes \Gamma$ and $L^\infty(Y) \rtimes \Lambda$ are isomorphic and have a unique Cartan subalgebra up to unitary conjugacy, then the actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are orbit equivalent.

1.4 Nonsingular countable equivalence relations

Let us start with the definition of a nonsingular countable equivalence relation.

Definition 1.4.1. Let (X, μ) be a standard probability space. A *nonsingular countable equivalence relation* $\mathcal{R}$ on (X, μ) is an equivalence relation such that

- $\mathcal{R} \subset X \times X$ is a Borel subset;
- the $\mathcal{R}$-equivalence class of x is countable, for almost every $x \in X$;
- every Borel automorphism $\varphi : X \to X$ such that $\mathrm{graph}(\varphi) \subset \mathcal{R}$ is nonsingular.

Let $U \subset X$ be a measurable subset, then we write $\mathcal{R}|_U$ for the nonsingular countable equivalence relation given by $\mathcal{R} \cap (U \times U)$. We call $\mathcal{R}|_U$ the *restriction* of $\mathcal{R}$ to U. We identify two nonsingular countable equivalence relations $\mathcal{R}$ and $\mathcal{R}'$ on (X, μ) whenever $\mathcal{R}|_U = \mathcal{R}'|_U$ for some conegligible subset $U \subset X$.

Every group action $\Gamma \curvearrowright (X, \mu)$ gives rise to a nonsingular countable equivalence relation in the following way:

$$\mathcal{R}(\Gamma \curvearrowright X) := \{(x, g \cdot x) \mid g \in \Gamma, x \in X\}.$$

This equivalence relation is called the *orbit equivalence relation* of $\Gamma \curvearrowright (X, \mu)$. The following theorem shows us that all nonsingular countable equivalence relations are of this form.

Theorem 1.4.2 ([FM77, Theorem 1])**.** *Let* $\mathcal{R}$ *be a nonsingular countable equivalence relation on a standard probability space* (X, μ)*. Then there exists a countable discrete group* Γ *and a group action* $\Gamma \curvearrowright (X, \mu)$ *satisfying* $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X)$*, up to conegligible restriction.*

Let $\mathcal{R}$ be a nonsingular countable equivalence relation on a standard probability space (X, μ). We denote by $[x]$ or $[x]_\mathcal{R}$ the equivalence class of $x \in X$. Let

$U \subset X$ be a measurable subset, we define the $\mathcal{R}$-*saturation* $[U]_{\mathcal{R}}$ of U by $\{x \in X \mid (x,y) \in \mathcal{R} \text{ for some } y \in U\}$. Furthermore we denote by $[\mathcal{R}]$ the *full group* of $\mathcal{R}$, i.e. the group of all nonsingular automorphisms φ of X such that the graph of φ is contained in $\mathcal{R}$. From Theorem 1.4.2, we see that there always exists a sequence $(\varphi_n)_n \in [\mathcal{R}]$ satisfying $\mathcal{R} = \bigcup_{n\in\mathbb{N}} \mathrm{graph}(\varphi_n)$.

Next to the full group of $\mathcal{R}$, we also have the full pseudogroup of $\mathcal{R}$. Let us recall what this is. We denote by $\mathrm{PAut}(X)$ the set of all partial nonsingular automorphisms φ of X, i.e. isomorphisms $\varphi : Y \to Z$, where $Y, Z \subset X$ are Borel subsets. The composition of two partial automorphisms is defined as follows: if $\alpha, \beta \in \mathrm{PAut}(X)$ are given by $\alpha : Y \to Z$ and $\beta : V \to W$ for Borel subsets $Y, Z, V, W \subset X$, then the composition $\alpha \circ \beta \in \mathrm{PAut}(X)$ is defined by $x \mapsto \alpha(\beta(x))$ for all $x \in \beta^{-1}(W \cap Y)$. Finally, the *full pseudogroup* $[[\mathcal{R}]]$ of $\mathcal{R}$ is the set of all $\varphi \in \mathrm{PAut}(X)$ such that the graph of φ is contained in $\mathcal{R}$.

As with group actions, one has different ways of expressing when two equivalence relations are ‘equivalent’.

Definition 1.4.3. Let $\mathcal{R}$ be a nonsingular countable equivalence relation on (X,μ) and let $\mathcal{R}'$ be a nonsingular countable equivalence relation on (X',μ'). The equivalence relations $\mathcal{R}$ and $\mathcal{R}'$ are called

- *isomorphic*, if there exist a nonsingular isomorphism $\Delta : X \to X'$ such that $\Delta([x]) = [\Delta(x)]$ for almost every $x \in X$;
- *stably isomorphic*, if there exist nonnegligible Borel subsets $Z \subset X$ and $Z' \subset X'$ that meet the class of almost every point and there exists a nonsingular isomorphism $\Delta : Z \to Z'$ such that $\Delta([x] \cap Z) = [\Delta(x)] \cap Z'$ for almost every $x \in Z$.

Note that these two notions are compatible with the notions of orbit equivalence and stable orbit equivalence for group actions in the sense that $\mathcal{R}(\Gamma \curvearrowright X)$ and $\mathcal{R}(\Lambda \curvearrowright Y)$ are (stably) isomorphic if and only if $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are (stably) orbit equivalent.

As with group actions, we also have the notion of ergodicity for equivalence relations.

Definition 1.4.4. Let $\mathcal{R}$ be a nonsingular countable equivalence relation on a standard probability space (X,μ). We call $\mathcal{R}$ *ergodic* if every $\mathcal{R}$-invariant Borel subset of X has measure 0 or 1.

Clearly $\mathcal{R}(\Gamma \curvearrowright X)$ is ergodic if and only if $\Gamma \curvearrowright X$ is ergodic. So by Theorem 1.4.2 we have that nonsingular countable equivalence relations are ergodic if and only if every $\mathcal{R}$-invariant element of $L^\infty(X,\mu)$ is constant almost everywhere.

An important class of nonsingular countable equivalence relations are the probability measure preserving (pmp) countable equivalence relations. To introduce them, we need the left and right counting measures.

Definition 1.4.5 ([FM77])**.** Let $\mathcal{R}$ be a nonsingular countable equivalence relation on (X,μ). We define the left and right counting measures on $\mathcal{R}$ as follows:

$$\mu_l(A) := \int_X \#\{y \in X \mid (x,y) \in A\} d\mu(x);$$

$$\mu_r(A) := \int_X \#\{x \in X \mid (x,y) \in A\} d\mu(y).$$

The equivalence relation $\mathcal{R}$ is said to preserve the measure μ if $\mu_l = \mu_r$. In that case, we denote μ_l by $\mu^{(1)}$.

For every n one can 'extend' a pmp countable equivalence relation $\mathcal{R}$ to $\mathcal{R}^{(n)}$ in the following way:

$$\mathcal{R}^{(n)} := \{(x_1, \ldots, x_{n+1}) \mid (x_1, x_2) \in \mathcal{R}, \ldots, (x_n, x_{n+1}) \in \mathcal{R}\}.$$

We also have the following measures on $\mathcal{R}^{(n)}$:

$$\mu_i^{(n)}(A) := \int_X \#\{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) | (x_1, \ldots, x_{n+1}) \in A\} d\mu(x_i).$$

Since $\mathcal{R}$ is pmp, we have that the measures $\mu_i^{(n)}$ do not depend on the choice of $i \in \{1, \ldots, n+1\}$.

Lemma 1.4.6 below gives another characterisation for being pmp. With this characterisation, we see that $\mathcal{R}(\Gamma \curvearrowright X)$ is pmp if and only if $\Gamma \curvearrowright X$ is pmp.

Lemma 1.4.6 ([FM77, Corollary 1])**.** *Let $\mathcal{R}$ be a nonsingular countable equivalence relation on (X,μ). The equivalence relation $\mathcal{R}$ is pmp if and only if every element $\varphi \in [[\mathcal{R}]]$ is probability measure preserving.*

Let us now introduce hyperfiniteness for nonsingular countable equivalence relations.

Definition 1.4.7. Let $\mathcal{R}$ be a nonsingular countable equivalence relation on a standard probability space (X,μ). Then $\mathcal{R}$ is called *hyperfinite* if it is, up to conegligible restriction, the union of an increasing sequence of finite (i.e. every class is finite) nonsingular equivalence relations.

The following theorem gives us many examples of hyperfinite equivalence relations.

Theorem 1.4.8 ([OW80, Theorem 6]). *If $\Gamma \curvearrowright (X, \mu)$ is an action of an amenable group Γ, then $\mathcal{R}(\Gamma \curvearrowright X)$ is hyperfinite.*

We end this section by explaining the notion of index for inclusions of nonsingular countable equivalence relations. To that end, let $\mathcal{S}$ be a subequivalence relation of a nonsingular countable equivalence relation $\mathcal{R}$. Define the map $\mathcal{I} : X \to \mathbb{N} \cup \{+\infty\}$ where $\mathcal{I}(x)$ is equal to the number of $\mathcal{S}$-classes in $[x]_{\mathcal{R}}$. From the beginning of the proof of [FSZ89, Lemma 1.1], we get that $\mathcal{I}$ is a Borel map. We call $\mathcal{I}$ the *index function* of the inclusion $\mathcal{S} \subset \mathcal{R}$ and say that the inclusion $\mathcal{S} \subset \mathcal{R}$ has *finite index*, if $\mathcal{I}(x) < +\infty$ for almost every $x \in X$.

If $\mathcal{I} \equiv n$ for some $n \in \mathbb{N} \cup \{+\infty\}$, we say that $\mathcal{S} \subset \mathcal{R}$ has index n and we write $[\mathcal{R} : \mathcal{S}] = n$. Note that, whenever $\mathcal{R}$ is ergodic, the index function is always essentially constant and hence the index $[\mathcal{R} : \mathcal{S}]$ is defined for every subequivalence relation $\mathcal{S}$.

The following result can be found in between the lines of [FSZ89, Lemma 1.1 and Lemma 1.3]. For the convenience of the reader, we include a proof for it.

Lemma 1.4.9. *Let $\mathcal{R}$ be a nonsingular countable Borel equivalence relation on a standard probability space (X, μ). Let $\mathcal{S}$ be a subequivalence relation of $\mathcal{R}$ and let n be a nonzero positive integer satisfying $n \leq \mathcal{I}(x)$ for almost every x. Then there exist $\varphi_i \in [[\mathcal{R}]]$ for $1 \leq i \leq n$ and a nonnegligible Borel subset $U \subset X$ such that*

- $\mathrm{dom}(\varphi_i) = U$ *for every* $1 \leq i \leq n$*;*
- $[\varphi_1(x)]_{\mathcal{S}}, \ldots, [\varphi_n(x)]_{\mathcal{S}}$ *are disjoint for almost every* $x \in U$.

Proof. By Theorem 1.4.2, we may choose an action of a countable group $G = \{g_0 := e, g_1, \ldots\}$ on X such that $\mathcal{R} = \mathcal{R}(G \curvearrowright X)$. Define for $1 \leq i \leq n$ and $m \in \mathbb{N}$ inductively ϕ_i and $U_{i,m} \subset X$ as follows:

- $\phi_1 := \mathrm{id}$, $U_{1,0} := X$ and $U_{1,m} := \emptyset$ when $m > 0$;
- for $i > 1$, $U_{i,m} := \{x \mid m = \inf\{k \mid g_k \cdot x \notin \bigcup_{j=1}^{i-1} [\phi_j(x)]_{\mathcal{S}}\}\}$;
- for $i > 1$, $\phi_i(x) := g_m \cdot x$, whenever $x \in U_{i,m}$.

By construction, we have that $\mathrm{graph}(\phi_i) \subset \mathcal{R}$ for every $1 \leq i \leq n$. Also by construction, we have that the classes $[\phi_1(x)]_{\mathcal{S}}, \ldots, [\phi_n(x)]_{\mathcal{S}}$ are disjoint for almost every $x \in X$. However, it need not be true that the maps ϕ_i are injective.

Let us now restrict the maps ϕ_i to some nonnegligible $U \subset X$ such that $\phi_i|_U$ is injective for every i. In order to do this, note that ϕ_i is injective on $U_{i,m}$ for every $m \in \mathbb{N}$. Since $\{U_{i,m}\}_m$ partitions X for every i, we can find m_i such that $U := \bigcap_{i=1}^n U_{i,m_i}$ is nonnegligible. Write $\varphi_i := \phi_i|_U$ for $1 \leq i \leq n$. Then the φ_i have the required properties. $\square$

1.5 The type of an ergodic equivalence relation

In this section, we introduce the type of an ergodic equivalence relation, see e.g. [DS09, Proposition 18]. To that end, let $\mathcal{R}$ be an ergodic nonsingular countable equivalence relation on a standard probability space (X, μ). The *Radon-Nikodym 1-cocycle* of $\mathcal{R}$ (see [FM77, Proposition 2.2]) is the $\mu^{(1)}$-a.e. uniquely defined Borel map $\omega : \mathcal{R} \to \mathbb{R}$ such that

$$\omega(\varphi(x), x) = \log\Big(\frac{d\mu \circ \varphi}{d\mu}(x)\Big) \text{ for all } \varphi \in [[\mathcal{R}]] \text{ and almost every } x \in \operatorname{dom} \varphi \,.$$

Note that ω satisfies the 1-cocycle relation $\omega(x, z) = \omega(x, y) + \omega(y, z)$ for $\mu^{(2)}$-a.e. $(x, y, z) \in \mathcal{R}^{(2)}$. One then defines the *Maharam extension* ([Ma63]) $\widetilde{\mathcal{R}}$ of $\mathcal{R}$ as the equivalence relation on $(X \times \mathbb{R}, \mu \times \exp(-t)dt)$ defined by

$$(x, t) \sim (y, s) \quad \text{if and only if} \quad (x, y) \in \mathcal{R} \text{ and } t - s = \omega(x, y).$$

Note that $\mu \times \exp(-t)dt$ is an infinite invariant measure for $\widetilde{\mathcal{R}}$. Denote the von Neumann algebra of all $\widetilde{\mathcal{R}}$-invariant functions in $L^\infty(X \times \mathbb{R})$ by $L^\infty(X \times \mathbb{R})^{\widetilde{\mathcal{R}}}$. Since $\mathcal{R}$ was assumed to be ergodic, the action of $\mathbb{R}$ on $L^\infty(X \times \mathbb{R})^{\widetilde{\mathcal{R}}}$ given by translation of the second variable, is also ergodic. Depending on how this action of $\mathbb{R}$ looks like, we define as follows the type of $\mathcal{R}$.

- I or II, if the action is conjugate with $\mathbb{R} \curvearrowright \mathbb{R}$;
- III_λ $(0 < \lambda < 1)$, if the action is conjugate with $\mathbb{R} \curvearrowright \mathbb{R}/\mathbb{Z}\log(\lambda)$;
- III_1, if the action is on one point ;
- III_0, if the action is properly ergodic, i.e. is ergodic and has orbits of measure zero.

Remark. *Let $\mathcal{R}$ be a nonsingular countable equivalence relation on a standard probability space (X, μ). Denote by $L(\mathcal{R})$ the von Neumann algebra associated with $\mathcal{R}$ ([FM77b]). It turns out that the von Neumann algebra $L(\mathcal{R})$ is a factor if and only if $\mathcal{R}$ is ergodic ([FM77b, Proposition 2.9]). All von Neumann algebra factors can be classified into types I, II and III ([MvN36]). Using the modular theory of Connes and Takesaki, all factors of type III can furthermore* *be classified into types III$_\lambda$ with $0 \leq \lambda \leq 1$ ([Tak03]). It turns out that the type of an ergodic nonsingular equivalence relation $\mathcal{R}$ is the same as the type of the factor $L(\mathcal{R})$. To see this, denote by φ the normal semifinite faithful state on $L(\mathcal{R})$ that is induced by μ. Furthermore denote by $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ its modular automorphism group ([Tak03, Definition VIII.1.3]). There is a canonical identification $L(\widetilde{\mathcal{R}}) \cong L(\mathcal{R}) \rtimes_{\sigma^\varphi} \mathbb{R}$. Under this identification, the dual action of $\mathbb{R}$ on $L(\mathcal{R}) \rtimes_{\sigma^\varphi} \mathbb{R}$ corresponds to the action of $\mathbb{R}$ on $L(\widetilde{\mathcal{R}})$ that we defined above. Also, the center of $L(\mathcal{R}) \rtimes_{\sigma^\varphi} \mathbb{R}$ corresponds to $L^\infty(X \times \mathbb{R})^{\widetilde{\mathcal{R}}}$. Altogether, following [Tak03, Definition XII.1.5], we have that the type of the equivalence relation $\mathcal{R}$ indeed coincides with the type of the factor $L(\mathcal{R})$.*

Note that any pmp equivalence relation is not of type III. We refer to [DS09] for examples of ergodic countable equivalence relations that are of type III (see Example 17(i), 17(ii) and 19).

The following is a well known result for which we could not find a reference. We provide a proof for the convenience of the reader.

Lemma 1.5.1. *The type of an ergodic nonsingular countable equivalence relation is preserved under taking stable isomorphisms.*

Proof. Clearly, the type of an ergodic nonsingular countable equivalence relation is preserved under taking isomorphisms. It rests to show that it is also preserved under taking restrictions. So let $\mathcal{R}$ be an ergodic nonsingular countable equivalence relation on a standard probability space (X, μ) and let $U \subset X$ be a nonnegligible Borel subset. Using the ergodicity of $\mathcal{R}$ we have that

$$\alpha : L^\infty(X \times \mathbb{R})^{\widetilde{\mathcal{R}}} \to L^\infty(U \times \mathbb{R})^{\widetilde{\mathcal{R}}|_{U\times\mathbb{R}}} : F \to F|_{U\times\mathbb{R}}$$

is an $\mathbb{R}$-equivariant isomorphism. Furthermore $\widetilde{\mathcal{R}}|_{U\times\mathbb{R}} = \widetilde{\mathcal{R}|_U}$. Hence $\mathcal{R}$ and $\mathcal{R}|_U$ indeed have the same type. $\square$

The following lemma gives a sufficient condition for an ergodic equivalence relation to be of type III$_\lambda$. The result is known to experts, but we could not find a reference for it. Again a proof is provided for the convenience of the reader.

Lemma 1.5.2. *Let $\mathcal{R}$ be an ergodic nonsingular countable equivalence relation on a standard probability space (X, μ). Denote by ω its Radon-Nikodym 1-cocycle. If the essential image* $\mathrm{Im}(\omega)$ *of ω equals* $\log(\lambda)\mathbb{Z}$ *for some* $0 < \lambda < 1$ *and if the kernel* $\mathrm{Ker}(\omega)$ *of ω is an ergodic equivalence relation, then $\mathcal{R}$ is of type III$_\lambda$.*

Proof. Since $\mathrm{Ker}(\omega)$ is an ergodic equivalence relation on (X,μ), we have

$$L^\infty(X\times\mathbb{R})^{\widetilde{\mathcal{R}}} \subset L^\infty(X\times\mathbb{R})^{\mathrm{Ker}(\omega)} = 1\otimes L^\infty(\mathbb{R})\ .$$

For a given $F\in L^\infty(\mathbb{R})$, we have that $1\otimes F$ is $\widetilde{\mathcal{R}}$-invariant if and only if F is invariant under translation by the essential image of ω. So,

$$L^\infty(X\times\mathbb{R})^{\widetilde{\mathcal{R}}} = 1\otimes L^\infty(\mathbb{R}/\log(\lambda)\mathbb{Z})\ .$$

□

1.6 Hilbert bimodules

We start by giving the definition of a Hilbert bimodule.

Definition 1.6.1 (see e.g. [Co94]). Let (M,τ_M) and (N,τ_N) be tracial von Neumann algebras.

1. A *left M-module* ${}_M\mathcal{H}$ is a Hilbert space $\mathcal{H}$ equipped with a normal unital homomorphism $\pi_l : M\to \mathrm{B}(\mathcal{H})$;
2. A *right N-module* $\mathcal{H}_N$ is a Hilbert space $\mathcal{H}$ equipped with a normal unital anti-homomorphism $\pi_r : N\to \mathrm{B}(\mathcal{H})$ (i.e. a normal unital representation of the opposite algebra N^{op});
3. An *M-N-bimodule* ${}_M\mathcal{H}_N$ is a Hilbert space which is both a left M-module as a right N-module, such that the representations π_l and π_r commute.

Let $\mathcal{H}$ be an M-N-bimodule, for $x\in M, y\in N$ and $\xi\in\mathcal{H}$, we write $x\xi y$ instead of $\pi_l(x)\pi_r(y)(\xi)$.

If ${}_M\mathcal{H}_N$ is an M-N-bimodule, the *contragredient bimodule* ${}_N\overline{\mathcal{H}}_M$ is defined on the conjugate Hilbert space $\overline{\mathcal{H}}=\mathcal{H}^*$ with bimodule actions given by

$$x\cdot\overline{\xi} = \overline{\xi x^*} \text{ and } \overline{\xi}\cdot y = \overline{y^*\xi}.$$

Let us give an easy example of an M-M-bimodule, called the *identity bimodule* (see e.g. [Co94]). Denote by $L^2(M)$ the completion of M equipped with the inner product $\langle x,y\rangle = \tau(xy^*)$ and denote the associated norm by $||\cdot||_2$, so $||x||_2^2=\tau(xx^*)$. An element x of M can be viewed as a vector of $L^2(M)$ in which case it will be denoted by $\hat{x}$. For $x,y\in M$ we set

$$\pi_l(x)\hat{y} = \widehat{xy} \text{ and } \pi_r(x)\hat{y} = \widehat{yx}.$$

Since $||\widehat{xy}||_2^2 = \tau(xyy^*x^*) = \tau(y^*x^*xy) \leq \tau(y^*||x^*x||y) = ||x||^2||y||_2^2$, we have that $\pi_l(x)$ extends to an element of $\mathrm{B}(L^2(M))$. Similarly $\pi_r(x)$ extends to an element of $\mathrm{B}(L^2(M))$. We also have that the representations π_l and π_r commute. Actually, we even have that $\pi_l(M)' = \pi_r(M)$, see e.g. [JS97, Lemma 1.2.4.(1)]. In this way, we can view $L^2(M)$ as an M-M-bimodule.

From $L^2(M)$, one can build many examples of right M-modules. Let p be a *projection* (i.e. $p^2 = p = p^*$) in $\mathrm{B}(l^2(\mathbb{N}))\overline{\otimes}M$. Then $p(l^2(\mathbb{N}) \otimes L^2(M))$ is a right M-module in the obvious way. In a similar way, one can build many examples of left M-modules.

The following lemma is a well known result saying that all right M-modules are of this special form. Recall that two projections p and q in a von Neumann algebra are *equivalent* ([MvN36]), if there exists an element v in that von Neumann algebra satisfying $p = v^*v$ and $q = vv^*$.

Proposition 1.6.2 ([JS97, Theorem 2.2.2])**.** *Let (M, τ) be a tracial von Neumann algebra and let $\mathcal{H}$ be a countably generated right M-module. Then there exists a projection $p \in M^\infty := \mathrm{B}(l^2(\mathbb{N}))\,\overline{\otimes}\,M$, which can be taken diagonal, such that $\mathcal{H}$ and $p(l^2(\mathbb{N}) \otimes L^2(M))$ are isomorphic as right M-modules. Moreover, this correspondence defines a bijection between the class of countably generated right M-modules, up to isomorphism, and the set of equivalence classes of projections in $\mathrm{B}(l^2(\mathbb{N}))\,\overline{\otimes}\,M$.*

Denote by $\mathrm{B}(l^2(\mathbb{N}))^+ = \{x \in \mathrm{B}(l^2(\mathbb{N})) \mid \langle x\xi, \xi\rangle \geq 0 \text{ for every } \xi \in l^2(\mathbb{N})\}$ the set of all positive operators in $\mathrm{B}(l^2(\mathbb{N}))$. Now define the (infinite) trace

$$\mathrm{Tr} : \mathrm{B}(l^2(\mathbb{N}))^+ \to [0, +\infty] : x \mapsto \sum_{n\in\mathbb{N}} \langle xe_n, e_n\rangle,$$

where $(e_n)_n$ denotes the canonical orthonormal basis of $l^2(\mathbb{N})$. Following the notation of the previous proposition, since $(\mathrm{Tr} \otimes \tau_M)(p)$ is an invariant for the equivalence class of p, it is also an invariant for the isomorphism class of the right M-module $\mathcal{H}$. This invariant is called the *right dimension* of $\mathcal{H}$ and is denoted by $\dim_{-M}(\mathcal{H})$. All this was already known to Murray and von Neumann ([MvN36]). By considering a left M-module as a right M^{op}-module, we can also define the *left dimension* $\dim_{M-}(\mathcal{H})$ of a left M-module $\mathcal{H}$. Moreover, a bimodule ${}_M\mathcal{H}_N$ is said to have *finite index* when the dimension of ${}_M\mathcal{H}$ and $\mathcal{H}_N$ are both finite. Also, we call an M-N-bimodule *bifinite* if it is finitely generated both as a left Hilbert M-module and a right Hilbert N-module. Finally, if (M, τ) is a tracial von Neumann algebra and N is a von Neumann subalgebra of M, then we define the *Jones index* $[M : N]$ of the inclusion as $\dim_{\text{-}N} L^2(M)$, see [Jo83].

We conclude this section by introducing the notion of stable isomorphism between II_1 factors and explaining its connection with finite index bimodules, see e.g. [Po86]. Whenever M is a II_1 factor and $t > 0$, we denote by M^t the *amplification* of M. Up to unitary conjugacy, M^t is defined as $p(\mathrm{M}_n(\mathbb{C}) \otimes M)p$ where p is a projection satisfying $(\mathrm{Tr} \otimes \tau)(p) = t$. Two II_1 factors M and N are called *stably isomorphic* if there exists a $t > 0$ such that $M \cong N^t$. The existence of a finite index bimodule ${}_M\mathcal{H}_N$ introduces a link between the von Neumann algebras M and N that generalizes the notion of a stable isomorphism. Indeed, let M and N be II_1 factors and let $\alpha : M \to pN^n p$ be an isomorphism. Then ${}_M\mathcal{H}(\alpha)_N$ given by $\mathcal{H}(\alpha) = p(\mathbb{C}^n \otimes L^2(N))$ and $x\xi y = \alpha(x)\xi y$ is an M-N-bimodule with $\dim_{-N}(\mathcal{H}) = (\mathrm{Tr} \otimes \tau_N)(p)$ and $\dim_{M-}(\mathcal{H}) = 1/(\mathrm{Tr} \otimes \tau_N)(p)$.

1.7 Connes tensor products and weak containment

Let (N, τ_N) be a tracial von Neumann algebra. Let $\mathcal{H}$ be a right N-module and let $\mathcal{K}$ be a left N-module. We call a vector $\xi \in \mathcal{H}$ *right bounded*, if there exists $c > 0$ such that $||\xi x|| \leq c||x||_2$, for all $x \in N$. In that case we define the bounded linear operator $L_\xi : L^2(N) \to \mathcal{K}$ as

$$L_\xi(\hat{x}) = \xi x \text{ for every } x \in N.$$

We also denote by $\mathcal{H}_0$ the vector space of all right bounded vectors of $\mathcal{H}$. We have that $\mathcal{H}_0$ is dense in $\mathcal{H}$, see [Co94, Proposition 6a]. We also have the following lemma, see e.g. [Fa09, Lemma 3.5].

Lemma 1.7.1. *We have that $L^2(N)_0 = \hat{N}$.*

On the algebraic tensor product $\mathcal{H}_0 \odot \mathcal{K}$, we define the inner product

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle = \langle (L^*_{\xi'} L_\xi)\eta, \eta' \rangle.$$

Note that this makes sense, since $L^*_{\xi'} L_\xi$ is a bounded linear operator on $L^2(N)$ that commutes with the right N-action, and hence must be an element of N. The *Connes tensor product* $\mathcal{H} \otimes_N \mathcal{K}$ (see Appendix B.δ of [Co94]) is then defined as the completion of $(\mathcal{H}_0 \odot \mathcal{K})/N_{\langle \cdot,\cdot \rangle}$, where $N_{\langle \cdot,\cdot \rangle} := \{\zeta \in \mathcal{H}_0 \odot \mathcal{K} \mid \langle \zeta, \zeta \rangle = 0\}$.

The Connes tensor product $\mathcal{H} \otimes_N \mathcal{K}$ can also be obtained by looking at left bounded vectors of $\mathcal{K}$. We call a vector $\eta \in \mathcal{K}$ *left bounded*, if there exists $c > 0$ such that $||x\eta|| \leq c||x||_2$, for all $x \in N$. In that case we can define a bounded linear operator $R_\eta : L^2(N) \to \mathcal{K}$ by

$$R_\eta(\hat{x}) = x\eta \text{ for every } x \in N.$$

We denote by ${}_0\mathcal{K}$ the vector space of all left bounded vectors of $\mathcal{K}$. Also here ${}_0\mathcal{K}$ is dense in $\mathcal{K}$ and ${}_0L^2(N) = \hat{N}$. On the algebraic tensor product $\mathcal{H} \odot {}_0\mathcal{K}$, we define the inner product

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle = \langle \xi (J R_\eta^* R_{\eta'} J), \xi' \rangle,$$

where $J : \hat{x} \mapsto \widehat{x^*}$ is the canonical anti-unitary on $L^2(N)$. The Connes tensor product $\mathcal{H} \otimes_N \mathcal{K}$ is equivalently defined as the completion of $(\mathcal{H} \odot {}_0\mathcal{K})/N_{\langle \cdot,\cdot \rangle}$, where $N_{\langle \cdot,\cdot \rangle} := \{\zeta \in \mathcal{H} \odot {}_0\mathcal{K} \mid \langle \zeta, \zeta \rangle = 0\}$ (see [Po86]).

The following is an important property of the Connes tensor product. It is easily verified.

Proposition 1.7.2. *Let (N, τ_N) be a tracial von Neumann algebra. Let $\mathcal{H}$ be a right N-module and let $\mathcal{K}$ be a left N-module. Inside $\mathcal{H} \otimes_N \mathcal{K}$, we have that*

- *$\xi x \otimes \eta = \xi \otimes x\eta$ for every $\xi \in \mathcal{H}_0$, $\eta \in \mathcal{K}$ and $x \in N$;*
- *$\xi x \otimes \eta = \xi \otimes x\eta$ for every $\xi \in \mathcal{H}$, $\eta \in {}_0\mathcal{K}$ and $x \in N$.*

Proof. Let $\xi, \xi' \in \mathcal{H}_0$, $\eta, \eta' \in \mathcal{K}$ and $x \in N$. We can make the following calculation

$$\begin{aligned} \langle \xi x \otimes \eta, \xi' \otimes \eta' \rangle &= \langle (L_{\xi'}^* L_{\xi x})\eta, \eta' \rangle = \langle (L_{\xi'}^* L_\xi x)\eta, \eta' \rangle \\ &= \langle (L_{\xi'}^* L_\xi) x\eta, \eta' \rangle = \langle \xi \otimes x\eta, \xi' \otimes \eta' \rangle. \end{aligned}$$

This shows that $\xi x \otimes \eta = \xi \otimes x\eta$ for every $\xi \in \mathcal{H}_0$, $\eta \in \mathcal{K}$ and $x \in N$. A similar argument can be used to prove the second statement. □

Let (M, τ_M), (N, τ_N) and (P, τ_P) be tracial von Neumann algebras. Let $\mathcal{H}$ be an M-N-bimodule and let $\mathcal{K}$ be an N-P-bimodule. The Hilbert space $\mathcal{H} \otimes_N \mathcal{K}$ is then an M-P-bimodule, where $x(\xi \otimes \eta)y = x\xi \otimes \eta y$ (see Theorem 13 of [Co94]). Whenever N is a II_1 factor, the left and right dimension of this bimodule can easily be computed from the left and right dimensions of $\mathcal{H}$ and $\mathcal{K}$. This is the content of the following proposition which can be found in [JS97] without proof. For the convenience of the reader, we provide a proof for this result.

Proposition 1.7.3. *Let (M, τ_M) and (P, τ_P) be tracial von Neumann algebras. Let (N, τ_N) be a II_1 factor. If $\mathcal{H}$ is a finite index M-N-bimodule and $\mathcal{K}$ is a finite index N-P-bimodule, then*

- $\dim_{-P}(\mathcal{H} \otimes_N \mathcal{K}) = \dim_{-N}(\mathcal{H}) \dim_{-P}(\mathcal{K})$*;*
- $\dim_{M-}(\mathcal{H} \otimes_N \mathcal{K}) = \dim_{M-}(\mathcal{H}) \dim_{N-}(\mathcal{K})$*.*

Proof. We restrict ourselves to proving the first equality, since the second equality can be proven analogously.

So let (P, τ_P) be tracial von Neumann algebra and let (N, τ_N) be a II_1 factor. Furthermore, let $\mathcal{H}$ be a right N-module with $\dim_{-N} \mathcal{H} < \infty$ and let $\mathcal{K}$ be an N-P-bimodule with $\dim_{-P} \mathcal{K} < \infty$. Using Proposition 1.6.2, we may assume that

$$\mathcal{H}_N = (\bigoplus_i p_i L^2(N))_N,$$

where $p_i \in N$ are projections satisfying $\dim_{-N}(\mathcal{H}) = \sum_i \tau_N(p_i)$. In this way, we have that

$$(\mathcal{H} \otimes_N \mathcal{K})_P = (\bigoplus_i p_i \mathcal{K})_P.$$

On the other hand, using Proposition 1.6.2 again, we may assume that

$$\mathcal{K}_P = p(l^2(\mathbb{N}) \otimes L^2(P))_P,$$

where $p \in \mathrm{B}(l^2(\mathbb{N})) \overline{\otimes} P$ is a projection satisfying $\dim_{-P}(\mathcal{K}) = (\mathrm{Tr} \otimes \tau_P)(p) < \infty$. But since $\mathcal{K}$ is also a left N-module, there exists a normal $*$-homomorphism $\psi : N \to pP^\infty p$ such that the left N-action on $p(l^2(\mathbb{N}) \otimes L^2(P))$ is given by $x \cdot \xi = \psi(x)\xi$ for every $x \in N$. In this way, we see that

$$\dim_{-P}(p_i \mathcal{K}) = (\mathrm{Tr} \otimes \tau_P)(\psi(p_i)).$$

Since N is a factor, we have that ψ is faithful (see [Tak79, Proposition II.3.12]). Hence $(\mathrm{Tr} \otimes \tau_P)(\psi(\cdot))/(\mathrm{Tr} \otimes \tau_P)(p)$ is a normal faithful tracial state on N. Since N is a II_1 factor, we have that τ_N is the unique normal faithful tracial state (see [Tak79, Theorem V.2.6]). Therefore

$$(\mathrm{Tr} \otimes \tau_P)(\psi(x)) = \tau_N(x)(\mathrm{Tr} \otimes \tau_P)(p) \text{ for every } x \in N.$$

Putting everything together, we see that

$$\begin{aligned}
\dim_{-P}(\mathcal{H} \otimes_N \mathcal{K}) &= \dim_{-P}(\bigoplus_i p_i \mathcal{K}) = \sum_i \dim_{-P}(p_i \mathcal{K}) \\
&= \sum_i (\mathrm{Tr} \otimes \tau_P)(\psi(p_i)) = \sum_i \tau_N(p_i)(\mathrm{Tr} \otimes \tau_P)(p) \\
&= \dim_{-N}(\mathcal{H}) \dim_{-P}(\mathcal{K}).
\end{aligned}$$

□

Let us now introduce the notion of weak containment for bimodules.

Definition 1.7.4 ([Po86]). Let (M, τ_M) and (N, τ_N) be tracial von Neumann algebras and let $\mathcal{H}$ and $\mathcal{K}$ be two M-N-bimodules. We say that $\mathcal{H}$ is *weakly contained in* $\mathcal{K}$, if for every $\xi \in \mathcal{H}$, every finite subset E of M, every finite subset F of N and every $\epsilon > 0$, there exist $\eta_1, \ldots, \eta_n \in \mathcal{K}$ such that

$$|\langle x\xi y, \xi\rangle - \sum_{i=1}^{n} \langle x\eta_i y, \eta_i\rangle| \leq \epsilon$$

for every $x \in E$ and $y \in F$. If $\mathcal{H}$ is weakly contained in $\mathcal{K}$, we write ${}_M\mathcal{H}_N \prec {}_M\mathcal{K}_N$.

We have the following result saying that weak containment is stable under taking Connes tensor products, see e.g. [An95, Lemma 1.7].

Lemma 1.7.5. *Let (M, τ_M), (N, τ_N) and (P, τ_P) be tracial von Neumann algebras and let $\mathcal{H}$ and $\mathcal{K}$ be two M-N-bimodules. If ${}_M\mathcal{H}_N \prec {}_M\mathcal{K}_N$, then $\mathcal{H} \otimes_N \mathcal{L}$ is weakly contained in $\mathcal{K} \otimes_N \mathcal{L}$ for every N-P-bimodule $\mathcal{L}$. The same holds for weak containment in the second variable.*

1.8 Semi-finite von Neumann algebras and the basic construction

In this section we introduce the basic construction of an inclusion of tracial von Neumann algebras. These von Neumann algebras will turn out to be semi-finite. Let us first explain what this means.

1.8.1 Semi-finite von Neumann algebras

We already came across the functional

$$\mathrm{Tr} : \mathrm{B}(l^2(\mathbb{N}))^+ \to [0, +\infty] : x \mapsto \sum_{n \in \mathbb{N}} \langle xe_n, e_n\rangle,$$

where $(e_n)_n$ denotes the canonical orthonormal basis of $l^2(\mathbb{N})$. It is called the trace of $\mathrm{B}(l^2(\mathbb{N}))$ and is the prime example of a normal faithful semi-finite trace. In general we have the following definition.

Definition 1.8.1. Let M be a von Neumann algebra. We call a map

$$\mathrm{Tr} : M^+ \to [0, +\infty]$$

a *trace* if it is a positive-linear functional satisfying $\operatorname{Tr}(uxu^*) = \operatorname{Tr}(x)$ for all $x \in M^+$ and all unitaries $u \in \mathcal{U}(M)$.

It is called *semi-finite* if $\mathfrak{p} = \{x \in M^+ \mid \operatorname{Tr}(x) < \infty\}$ generates the whole of M as a von Neumann algebra.

It is called *normal* if $\operatorname{Tr}(\sup_i x_i) = \sup_i \operatorname{Tr}(x_i)$ for every bounded increasing net $(x_i)_i$ in M^+.

Finally, it is called *faithful* if $\operatorname{Tr}(x) = 0$ if and only if $x = 0$, for $x \geq 0$.

If M comes equipped with a normal faithful semi-finite trace Tr, then we say that (M, Tr) is a *semi-finite* von Neumann algebra.

Let $\mathfrak{n} = \{x \in M \mid x^*x \in \mathfrak{p}\}$ and $\mathfrak{m} = \{\sum_{i=1}^n y_i^* x_i \mid x_i, y_i \in \mathfrak{n}\}$. Lemma VII.1.2 of [Tak03] says that Tr can be extended to a linear functional on $\mathfrak{m}$. This will be useful in the proof of the following proposition.

Proposition 1.8.2. *Let* (M, Tr) *be a semi-finite von Neumann algebra. Assume that* $\operatorname{Tr}(1) < \infty$*. Then* Tr *can be extended to the whole of* M *in such a way that* $\operatorname{Tr}(\cdot)/\operatorname{Tr}(1)$ *is a normal faithful tracial state on* M*.*

Proof. First of all, if $\operatorname{Tr}(1) < \infty$, then $\operatorname{Tr}(x) < \infty$ for every $x \in M^+$. This follows immediately from the following calculation:

$$\operatorname{Tr}(x) \leq \operatorname{Tr}(||x||1) = ||x|| \operatorname{Tr}(1) < \infty, \text{ for every } x \in M^+.$$

From this, one finds that $\mathfrak{n} = M$ and hence that $\mathfrak{m} = M$. Since Tr can be extended to $\mathfrak{m}$, we are done. ☐

Let (M, τ) be a tracial von Neumann algebra and let $\mathcal{H}_M$ be a right M-module. Denote by $\mathrm{B}(\mathcal{H}_M)$ the algebra of bounded operators on $\mathcal{H}$ that commute with the right action of M. By Proposition 1.6.2 we know that there exists a projection $p \in M^\infty$ such that $\mathcal{H}$ and $p(l^2(\mathbb{N}) \otimes L^2(M))$ are isomorphic as right M-modules. Furthermore, by Lemma 2.1.5 of [JS97], we have that $\mathrm{B}(p(l^2(\mathbb{N}) \otimes L^2(M))_M)$ is equal to $p(\mathrm{B}(l^2(\mathbb{N})) \otimes M)p$. Now note that $(\operatorname{Tr} \otimes \tau)$ is a normal faithful semi-finite trace on $p(\mathrm{B}(l^2(\mathbb{N})) \otimes M)p$ mapping the unit to $\dim_{-M}(\mathcal{H})$. Hence there exists a normal faithful semi-finite trace Tr on $\mathrm{B}(\mathcal{H}_M)$ satisfying $\operatorname{Tr}(1) = \dim_{-M}(\mathcal{H})$.

Definition 1.8.3. The normal faithful semi-finite trace Tr on $\mathrm{B}(\mathcal{H}_M)$ that is constructed above is called the *canonical trace* on $\mathrm{B}(\mathcal{H}_M)$.

We have the following nice formula for the canonical trace on $\mathrm{B}(\mathcal{H}_M)$. This result is well known to experts, but we could not find a reference for it.

Proposition 1.8.4. *Let (M,τ) be a tracial von Neumann algebra and let $\mathcal{H}_M$ be a right M-module. The canonical trace on $\mathrm{B}(\mathcal{H}_M)$ satisfies the equality*

$$\mathrm{Tr}(TS^*) = \tau_M(T^*S)$$

for all right M-modular bounded linear operators $T,S : L^2(M) \to \mathcal{H}$.

Proof. By Proposition 1.6.2, we may assume that

$$\mathcal{H} = p(l^2(\mathbb{N}) \otimes L^2(M)),$$

where p is a projection in $\mathrm{B}(l^2(\mathbb{N}))\,\overline{\otimes}\,M$. Let $T \in \mathrm{B}(L^2(M)_M, \mathcal{H}_M)$ and write $T\hat{1} = \sum_n \delta_n \otimes \xi_n \in l^2(\mathbb{N}) \otimes L^2(M)$. For $x \in M$, we have that

$$||T\hat{x}||^2 = \sum_n ||\xi_k x||_2^2 \leq ||T||^2 ||x||_2^2.$$

In particular $||\xi_n x||_2 \leq ||T||\ ||x||_2$. Therefore $\xi_n \in L^2(M)_0$ and hence $\xi_n = \widehat{x_n} \in \widehat{M}$ by Lemma 1.7.1.

Furthermore we have for $x, y_0, y_1, \ldots \in M$ that

$$\begin{aligned}
\langle \hat{x}, T^*(\sum_n \delta_n \otimes \widehat{y_n})\rangle &= \langle T\hat{x}, \sum_n \delta_n \otimes \widehat{y_n}\rangle \\
&= \langle \sum_n \delta_n \otimes \widehat{x_n x}, \sum_n \delta_n \otimes \widehat{y_n}\rangle \\
&= \sum_n \langle \widehat{x_n x}, \widehat{y_n}\rangle = \sum_n \langle \widehat{x}, \widehat{x_n^* y_n}\rangle \\
&= \langle \widehat{x}, \sum_n \widehat{x_n^* y_n}\rangle.
\end{aligned}$$

Hence $T^*(\sum_n \delta_n \otimes \widehat{y_n}) = \sum_n \widehat{x_n^* y_n}$ and therefore

$$T^*T = \sum_n x_n^* x_n \in M \text{ and } TT^* = [x_i x_j^*]_{i,j} \in \mathrm{B}(l^2(N))\,\overline{\otimes}\,M.$$

But then $(\mathrm{Tr}_{\mathrm{B}(l^2(\mathbb{N}))} \otimes \tau_M)(TT^*) = \tau_M(T^*T)$. Since $\mathrm{Tr}_{\mathrm{B}(l^2(\mathbb{N}))} \otimes \tau_M$ is precisely the canonical trace Tr on $\mathrm{B}(\mathcal{H}_M)$, we have that $\mathrm{Tr}(TT^*) = \tau_M(T^*T)$ for every $T \in \mathrm{B}(L^2(M)_M, \mathcal{H}_M)$.

By polarization, we also have that $\mathrm{Tr}(TS^*) = \tau(T^*S)$ for every $S, T \in \mathrm{B}(L^2(M)_M, \mathcal{H}_M)$. $\square$

To end this subsection, we introduce the L^2-space of a semi-finite von Neumann algebra (M, Tr) (see e.g. [Tak03]). Define

$$\langle x, y \rangle := \mathrm{Tr}(y^* x) \text{ for all } x, y \in \mathfrak{n}.$$

Since Tr is faithful, we have that this sesquilinear functional turns $\mathfrak{n}$ into a pre-Hilbert space. The completion of this pre-Hilbert space is exactly the L^2-space $L^2(M, \mathrm{Tr})$ of (M, Tr). Moreover, since $(ax)^*(ax) \leq ||a||^2 x^* x$ for every $a \in M$ and $x \in \mathfrak{n}$, we have that the left and right M-action on $\mathfrak{n}$ extend to $L^2(M, \mathrm{Tr})$ turning it into an M-M-bimodule.

1.8.2 The basic construction

Let (M, τ) be a tracial von Neumann algebra and let $N \subset M$ be a von Neumann subalgebra. A *conditional expectation* from M onto N is a linear map $E : M \to N$ satisfying

1. $E(M^+) \subset N^+$;
2. $E(y) = y$ for $y \in N$;
3. $E(y_1 x y_1) = y_1 E(x) y_2$ for $y_1, y_2 \in N$ and $x \in M$.

Now denote by e_N the orthogonal projection of $L^2(M)$ onto $L^2(N)$. The following lemma says that e_N induces a conditional expectation from M onto N. This result is stated in Subsection 3.1 of [JS97] without proof. For the convenience of the reader, we provide a proof of this well known result.

Lemma 1.8.5. *Let (M, τ) be a tracial von Neumann algebra and let $N \subset M$ be a von Neumann subalgebra. The projection e_N maps the dense subspace M of $L^2(M)$ into the subspace N. Let $E_N : M \to N$ denote the restriction $e_N|_M$, then E_N is a conditional expectation from M onto N. In fact, it is the unique conditional expectation from M onto N such that $\tau \circ E_N = \tau$.*

Proof. We first show that $e_N(M) \subset N$. Let $x \in M$ and $y_1, y_2 \in N$, then

$$\langle e_N(\hat{x}) y_1, \widehat{y_2} \rangle = \langle e_N(\hat{x}), \widehat{y_2 y_1^*} \rangle = \langle \hat{x}, \widehat{y_2 y_1^*} \rangle = \langle x \widehat{y_1}, \widehat{y_2} \rangle. \tag{1.1}$$

From this, we get that $e_N(\hat{x}) \in L^2(N)_0$. Using Lemma 1.7.1, we get that $e_N(\hat{x}) \in \widehat{N}$. Therefore the restriction $E_N = e_N|_M$ is a linear map from M to N. Clearly $E_N(y) = y$ for every $y \in N$. Furthermore since $L^2(N)$ is an N-N-subbimodule of $L^2(M)$, we also have that $E_N(y_1 x y_1) = y_1 E_N(x) y_2$ for every $y_1, y_2 \in N$ and $x \in M$. Let us now show that $E_N(M^+) \subset N^+$. For that, let $x \in M^+$ and $y \in N$. By Equation 1.1 we have that

$$\langle E_N(x)\hat{y}, \hat{y}\rangle = \langle x\hat{y}, \hat{y}\rangle \geq 0.$$

So indeed $E_N(M^+) \subset N^+$. Now we show that E_N is trace preserving. For every $x \in M$, we have that

$$\tau(E_N(x)) = \langle E_N(x)\hat{1}, \hat{1}\rangle = \langle x\hat{1}, \hat{1}\rangle = \tau(x).$$

Hence E_N is trace preserving. It remains to prove that E_N is the unique trace preserving conditional expectation from M onto N. To that end, let E be any trace preserving conditional expectation from M onto N. Then for $x \in M$ and $y \in N$, we have that

$$\tau((x - E(x))y) = \tau(E(x - E(x))y) = 0,$$

i.e. $\hat{x} - \widehat{E(x)}$ is orthogonal to $L^2(N)$. Hence, E has to be the orthogonal projection from $M \subset L^2(M)$ onto $N \subset L^2(N)$. In other words E and E_N must coincide. This ends the proof of the lemma. $\square$

Definition 1.8.6 (Jones, [Jo83])**.** Let (M, τ) be a tracial von Neumann algebra and $N \subset M$ a von Neumann subalgebra. *Jones' basic construction* $\langle M, e_N\rangle$ is the von Neumann algebra on $L^2(M)$ generated by M and e_N.

The following are some elementary properties of the basic construction, see e.g. page 484 of [BO08].

Proposition 1.8.7. *Let (M, τ) be a tracial von Neumann algebra and let $N \subset M$ be a von Neumann subalgebra. Then*

1. *$e_N x e_N = E_N(x)e_N$ for every $x \in M$;*
2. *$\langle M, e_N\rangle = \mathrm{B}(L^2(M)_N)$;*
3. *$\langle M, e_N\rangle = \overline{\mathrm{span}}^{w.o.}\{xe_N y \mid x, y \in M\}$.*

Since $\langle M, e_N\rangle = \mathrm{B}(L^2(M)_N)$ we have, following Definition 1.8.3, that $\langle M, e_N\rangle$ has a canonical normal faithful semi-finite trace Tr. Furthermore Tr satisfies $\mathrm{Tr}(1) = [M : N]$. The following proposition gives a nice formula for Tr. It can be found in [BO08] as Exercise F.6.

Proposition 1.8.8. *The canonical trace* Tr *on $\langle M, e_N\rangle$ satisfies* $\mathrm{Tr}(xe_N y) = \tau_M(xy)$ *for every $x, y \in M$.*

Proof. For every $x \in M$, define

$$L_x : L^2(N) \to L^2(M) : \xi \mapsto x\xi.$$

Then L_x is an element of $\mathrm{B}(L^2(N)_N, L^2(M)_N)$. Let $\xi \in L^2(N)$ and $\eta \in L^2(M)$, then

$$\begin{aligned}\langle \xi, L_x^*(\eta)\rangle &= \langle L_x\xi, \eta\rangle = \langle x\xi, \eta\rangle \\ &= \langle \xi, x^*\eta\rangle = \langle \xi, e_N(x^*\eta)\rangle.\end{aligned}$$

Therefore, we have that $L_x^* : L^2(M) \to L^2(N)$ satisfies $L_x^*(\eta) = e_N(x^*\eta)$. From this we find for every $x, y \in M$ that

$$L_x L_y^* = xe_N y^* \in \langle M, e_N\rangle \text{ and } L_y^* L_x = E_N(y^*x) \in N.$$

Using Lemma 1.8.4 yields $\mathrm{Tr}(xe_N y) = \tau_N(E_N(xy))$ for every $x, y \in M$. Since the conditional expectation E_N is trace preserving, we have found the desired formula. □

In the previous subsection, we introduced the notion of L^2-space of a semi-finite von Neumann algebra. In particular, we can define $L^2(\langle M, e_N\rangle, \mathrm{Tr})$. It turns out that $L^2(M) \otimes_N L^2(M)$ and $L^2(\langle M, e_N\rangle)$ are isomorphic as M-M-bimodules. To see this, define $\mathcal{M} := \mathrm{span}\{xe_N y \in \langle M, e_N\rangle \mid x, y \in M\}$. The map

$$T : \mathcal{M} \to L^2(M) \otimes_N L^2(M) : xe_N y \mapsto x \otimes y$$

can be extended to a unitary from $L^2(\langle M, e_N\rangle)$ onto $L^2(M) \otimes_N L^2(M)$. Indeed, we have that

$$\begin{aligned}\langle xe_N y, ze_N w\rangle &= \mathrm{Tr}((xe_N y)(ze_N w)^*) = \mathrm{Tr}(xE_N(yw^*)e_N z^*) \\ &= \tau_M(xE_N(yw^*)z^*) = \langle xE_N(yw^*), z\rangle \\ &= \langle x \otimes y, z \otimes w\rangle,\end{aligned}$$

for every $x, y, z, w \in M$. This unitary is clearly M-M-bimodular. Hence $L^2(M) \otimes_N L^2(M)$ and $L^2(\langle M, e_N\rangle)$ are indeed isomorphic as M-M-bimodules.

1.9 Amenability and relative amenability

Recall from before that a countable group Γ is called amenable if it admits a left invariant mean. It is a classical result that amenability of Γ can equivalently be expressed as a purely von Neumann algebraic property of $L(\Gamma)$. This is the content of the following proposition. Although a proof can be found in [Ho13, Theorem 7], we give a more direct proof for the convenience of the reader.

Proposition 1.9.1. *Let Γ be a countable group and let $M := L(\Gamma)$. Then Γ is amenable if and only if there exists a conditional expectation from* $\mathrm{B}(L^2(M))$ *onto M.*

Proof. Assume first that E is a conditional expectation from $\mathrm{B}(l^2(\Gamma))$ on $L(\Gamma)$. For every $f \in l^\infty(\Gamma)$, we denote by M_f the multiplication operator by f on $l^2(\Gamma)$. Set $m(f) := \tau_M(E(M_f))$. Since $u_g M_f u_g^* = M_{g\cdot f}$ for every $g \in \Gamma$, we see that

$$m(g \cdot f) = \tau_M(E(M_{g\cdot f})) = \tau_M(E(u_g M_f u_g^*))$$

$$= \tau_M(u_g E(M_f) u_g^*) = \tau_M(E(M_f)) = m(f).$$

Therefore m is a left invariant state on $l^\infty(\Gamma)$, and hence Γ is amenable.

Conversely, let Γ be amenable and let m be a left invariant mean on $l^\infty(\Gamma)$. Given $\xi, \eta \in l^2(\Gamma)$ and $T \in \mathrm{B}(l^2(\Gamma))$, we set

$$f^T_{\xi,\eta}(g) = \langle \xi, \rho(g) T \rho(g^{-1})\eta \rangle,$$

where ρ is the right regular representation of Γ. We have that

$$|f^T_{\xi,\eta}(g)| \leq ||T||\ ||\xi||\ ||\eta||.$$

Hence $f^T_{\xi,\eta}$ is a bounded function on Γ. Now define on $l^2(\Gamma)$ the following continuous sesquilinear functional:

$$(\xi, \eta) = m(f^T_{\xi,\eta}).$$

It follows that there exists a unique operator $E(T) \in \mathrm{B}(l^2(\Gamma))$ with

$$\langle \xi, E(T)\eta \rangle = m(f^T_{\xi,\eta}),$$

for every $\xi, \eta \in l^2(\Gamma)$. Furthermore, we see that

$$\langle \xi, \rho(g) E(T) \rho(g^{-1})\eta \rangle = \langle \rho(g^{-1})\xi, E(T)\rho(g^{-1})\eta \rangle$$

$$= m(f^T_{\rho(g^{-1})\xi, \rho(g^{-1})\eta})$$

$$= m(g \cdot f^T_{\xi,\eta}),$$

for every $\xi, \eta \in l^2(\Gamma)$ and $g \in \Gamma$. Since m is left invariant, we have that $m(g \cdot f^T_{\xi,\eta}) = m(f^T_{\xi,\eta})$. Hence

$$\langle \xi, \rho(g) E(T) \rho(g^{-1})\eta \rangle = m(g \cdot f^T_{\xi,\eta}) = m(f^T_{\xi,\eta}) = \langle \xi, E(T)\eta \rangle.$$

This shows that $E(T)$ is a bounded linear operator on $l^2(\Gamma)$ that commutes with the right $L(\Gamma)$-action. Hence $E(T) \in L(\Gamma)$. Thus so far we have found a linear map $E : \mathrm{B}(l^2(\Gamma)) \to L(\Gamma)$. We claim that E is a conditional expectation.

Let us show that E is positive. To that end, let $T \in \mathrm{B}(l^2(\Gamma))$ be a positive operator. Then

$$f^T_{\xi,\xi}(g) = \langle \xi, \rho(g)T\rho(g^{-1})\xi \rangle = \langle \rho(g^{-1})\xi, T\rho(g^{-1})\xi \rangle \geq 0,$$

for every $g \in \Gamma$ and $\xi \in l^2(\Gamma)$. So $f^T_{\xi,\xi}$ is a positive function and therefore $m(f^T_{\xi,\xi}) \geq 0$. But then

$$\langle \xi, E(T)\xi \rangle = m(f^T_{\xi,\xi}) \geq 0,$$

for every $\xi \in l^2(\Gamma)$. In other words, $E(T)$ is indeed positive.

Now fix $x \in L(\Gamma)$. We show that $E(x) = x$. For every $\xi, \eta \in l^2(\Gamma)$, we have that

$$f^x_{\xi,\eta} \equiv \langle \xi, x\eta \rangle.$$

Therefore $m(f^x_{\xi,\eta}) = \langle \xi, x\eta \rangle$ and so

$$\langle \xi, E(x)\eta \rangle = m(f^x_{\xi,\eta}) = \langle \xi, x\eta \rangle.$$

This shows that indeed $E(x) = x$ for every $x \in L(\Gamma)$.

Finally, fix $x, y \in L(\Gamma)$ and $T \in \mathrm{B}(l^2(\Gamma))$. It rests to show that $E(xTy) = xE(T)y$. However this follows immediately from the the fact that

$$\begin{aligned}\langle \xi, E(xTy)\eta \rangle &= m(f^{xTy}_{\xi,\eta}) = m(f^T_{x^*\xi,y\eta}) \\ &= \langle x^*\xi, E(T)y\eta \rangle \\ &= \langle \xi, xE(T)y\eta \rangle,\end{aligned}$$

for every $\xi, \eta \in l^2(\Gamma)$. □

This motivates the following definition.

Definition 1.9.2. We say that a tracial von Neumann algebra M is *amenable*, or *injective*, if there exists a conditional expectation from $\mathrm{B}(L^2(M))$ onto M.

As with amenability for groups, we also have many different characterisations for amenability in the von Neumann algebra case.

Theorem 1.9.3 ([Co75, Theorem 5.1] and [Po86, Theorem 3.1.2])**.** *Let (M, τ) be a tracial von Neumann algebra. The following conditions are equivalent*

1. *M is amenable;*

2. *There exists a hypertrace on $\mathrm{B}(L^2(M))$, i.e. a state ψ on $\mathrm{B}(L^2(M))$ such that $\psi(Tx) = \psi(xT)$ for every $x \in M$ and $T \in \mathrm{B}(L^2(M))$ and $\psi|_M = \tau$.*

3. *There exists a sequence $\xi_n \in L^2(M) \otimes L^2(M)$ such that $\langle x\xi_n, \xi_n\rangle \to \tau(x)$ and $||x\xi_n - \xi_n x||_2 \to 0$, for every $x \in M$.*

4. *${}_M L^2(M)_M$ is weakly contained in ${}_M(L^2(M) \otimes L^2(M))_M$.*

The next lemma will be useful for the proofs of both main results. It can be proven by slightly adapting the proof that Γ is amenable whenever $L(\Gamma)$ is amenable (see Proposition 1.9.1). For completeness, we give the argument completely.

Lemma 1.9.4. *Let Γ be a nonamenable countable group. Then $L(\Gamma)$ has no amenable direct summand.*

Proof. Assume, by way of reaching a contradiction, that $p \in L(\Gamma)$ is a nonzero central projection such that $pL(\Gamma)$ is amenable. Then there exists a $pL(\Gamma)$-central state ψ on $\mathrm{B}(pl^2(\Gamma))$. Now view $l^\infty(\Gamma) \subset \mathrm{B}(l^2(\Gamma))$ as diagonal multiplication operators, i.e. $(M_f\xi)(g) = f(g)\xi(g)$. We construct a state φ on $l^\infty(\Gamma)$ by $\varphi(f) = \psi(pM_f p)$. Then

$$\varphi(g \cdot f) = \psi(pM_{g\cdot f}p) = \psi(pu_g pM_f pu_g^* p) = \psi(pM_f p) = \varphi(f).$$

Hence φ is an invariant mean for Γ, and so Γ is amenable. We have arrived at a contradiction. □

Next to amenability, one also has the notion of relative amenability.

Definition 1.9.5 ([OP07, Theorem 2.1])**.** Let (M, τ) be a tracial von Neumann algebra and let $Q \subset M$ be a von Neumann subalgebra. Let $P \subset pMp$ be a von Neumann subalgebra, where $p \in M$ is a projection. We say that P is *amenable relative to Q inside M* if one of the following equivalent statements holds.

1. There exists a conditional expectation $E : p\langle M, e_Q\rangle p \to P$ such that the restriction $E|_{pMp}$ is equal to $E_P : pMp \to P$.

2. There exists a P-central state φ on $p\langle M, e_Q\rangle p$ such that $\varphi|_{pMp} = \tau_{pMp}$.

3. There exists a sequence $\xi_n \in L^2(p\langle M, e_Q\rangle p)$ such that $\langle x\xi_n, \xi_n\rangle \to \tau(x)$, for every $x \in pMp$, and $||y\xi_n - \xi_n y||_2 \to 0$, for every $y \in P$.

4. ${}_{pMp}L^2(pMp)_P$ is weakly contained in ${}_{pMp}(pL^2(M) \otimes_Q L^2(M)p)_P$.

Whenever M is amenable relative to Q inside M, we say that Q is *co-amenable* in M.

We have the following lemma showing that relative amenability is really a relative-version of amenability. It is a well known result for which we could not find a reference.

Lemma 1.9.6. *Let (M, τ) be a tracial von Neumann algebra and let $Q \subset M$ be a von Neumann subalgebra. Let $P \subset pMp$ be a von Neumann subalgebra, where $p \in M$ is a nonzero projection.*

1. *If P is amenable, then P is amenable relative to Q inside M.*
2. *If Q is amenable and P is amenable relative to Q inside M, then P is also amenable.*

Proof. To prove the first statement, we assume that P is amenable. By definition there exists a conditional expectation E from $\mathrm{B}(L^2(P))$ onto P. Denote by e_P the orthogonal projection of $L^2(M)$ onto $L^2(P)$. Under the identification of $e_P\,\mathrm{B}(L^2(M))e_P$ with $\mathrm{B}(L^2(P))$, we define the state

$$\varphi : p\langle M, e_Q\rangle p \to \mathbb{C}1 : x \mapsto \tau_{pMp}(E(e_P x e_P)).$$

Let $T \in p\langle M, e_Q\rangle p$ and $x \in P$. Then,

$$\begin{aligned}\varphi(xT) &= \tau_{pMp}(E(e_P xTe_P)) = \tau_{pMp}(E(xe_P Te_P)) \\ &= \tau_{pMp}(xE(e_P Te_P)) = \tau_{pMp}(E(e_P Te_P)x) \\ &= \tau_{pMp}(E(e_P Te_P x)) = \tau_{pMp}(E(e_P Txe_P)) \\ &= \varphi(Tx).\end{aligned}$$

So φ is a P-central state on $p\langle M, e_Q\rangle p$. Now let $x \in pMp$. Then we also have that

$$\begin{aligned}\varphi(x) &= \tau_{pMp}(E(e_P xe_P)) = \tau_{pMp}(E(e_P E_P(x)e_P)) \\ &= \tau_{pMp}(E(E_P(x))) = \tau_{pMp}(E_P(x)) \\ &= \tau_{pMp}(x).\end{aligned}$$

So $\varphi|_{pMp} = \tau_{pMp}$. Altogether, this shows that P is amenable relative to Q inside M.

To prove the second statement, assume that P is amenable relative to Q inside M and Q is amenable. Since Q is amenable, we have that

$$_Q L^2(Q)_Q \prec {}_Q L^2(Q) \otimes L^2(Q)_Q.$$

Taking the Connes tensor product on the left with $_{pMp} pL^2(M)_Q$ and on the right with $_Q L^2(M)p_{pMp}$, we find using Lemma 1.7.5 that

$$_{pMp} pL^2(M) \otimes_Q L^2(M)p_{pMp} \prec {}_{pMp} pL^2(M) \otimes L^2(M)p_{pMp}.$$

On the other hand, since P is amenable relative to Q inside M, we have that

$$_{pMp} L^2(pMp)_P \prec {}_{pMp} pL^2(M) \otimes_Q L^2(M)p_P.$$

Together, we find that

$$_{pMp} L^2(pMp)_P \prec {}_{pMp} pL^2(M) \otimes L^2(M)p_P.$$

In other words P is amenable relative to $\mathbb{C}1$ inside M. But then we have, by definition, a P-central state φ on $p\,\mathrm{B}(L^2(M))p$ such that $\varphi|_{pMp} = \tau_{pMp}$. The restriction $\varphi|_{\mathrm{B}(L^2(P))}$ is then a hypertrace on $\mathrm{B}(L^2(P))$, proving that P is amenable. □

1.10 Popa's intertwining-by-bimodules

An inclusion of von Neumann algebras $A \subset M$ is called *nonunital* whenever the unit of A does not coincide with the unit of M. In that case, we also call A a *nonunital von Neumann subalgebra* of M. In the next theorem we allow such nonunital inclusions. The result itself is called Popa's intertwining-by-bimodules theorem.

Theorem 1.10.1 ([Po03, Theorem 2.1 and Corollary 2.3])**.** *Let (M, τ) be a tracial von Neumann algebra and let $A, B \subset M$ be possibly nonunital von Neumann subalgebras. Denote their respective units by 1_A and 1_B. The following five conditions are equivalent:*

1. *$1_A L^2(M) 1_B$ admits a nonzero A-B-subbimodule that is finitely generated as a right B-module.*
2. *$1_A L^2(M) 1_B$ admits a nonzero A-B-subbimodule that has a finite right B-dimension.*
3. *There exist nonzero projections $p \in A$, $q \in B$, a normal unital $*$-homomorphism $\psi : pAp \to qBq$ and a nonzero partial isometry $v \in pMq$ such that $av = v\psi(a)$ for all $a \in pAp$.*

4. *There exists a nonzero projection $q \in M_n(\mathbb{C}) \otimes B$, a normal unital $*$-homomorphism $\psi : A \to q(M_n(\mathbb{C}) \otimes B)q$ and a nonzero partial isometry $v \in (M_{1,n}(\mathbb{C}) \otimes 1_A M)q$ such that $av = v\psi(a)$ for all $a \in A$.*

5. *There is no sequence of unitaries $u_n \in \mathcal{U}(A)$ satisfying $||E_B(xu_ny^*)||_2 \to 0$ for all $x, y \in 1_B M 1_A$.*

If one of these equivalent conditions holds, we write $A \prec_M B$ and say that A intertwines into B inside M.

The following lemma can be found in [Va08, Lemma 3.4] where the proof is left as an exercise. For the convenience of the reader we provide a proof here.

Lemma 1.10.2. *Let $A, B \subset (M, \tau)$ be, possibly nonunital, embeddings. Let $q_0 \in A$, $q_1 \in A' \cap 1_A M 1_A$, $p_0 \in B$ and $p_1 \in B' \cap 1_B M 1_B$ be nonzero projections.*

- *If $q_0 A q_0 \prec_M B$ or if $q_1 A \prec_M B$, then $A \prec_M B$;*
- *If $A \prec_M p_0 B p_0$ or if $A \prec_M p_1 B$, then $A \prec_M B$;*

Proof. The fact that $q_0 A q_0 \prec_M B$ and $A \prec_M p_0 B p_0$ both imply that $A \prec_M B$, follows immediately from the third characterisation in Theorem 1.10.1.

On the other hand, using the first characterisation in Theorem 1.10.1, we also see that $q_1 A \prec_M B$ and $A \prec_M p_1 B$ both imply that $A \prec_M B$. $\square$

The following lemma can be found in [Va08, Lemma 3.9].

Lemma 1.10.3. *Let $A, B \subset (M, \tau)$ be, possibly nonunital, embeddings.*

- *If $A \prec_M B$ and if $D \subset B$ is a unital finite index inclusion, then $A \prec_M D$;*
- *If $A \prec_M B$ and if $A \subset D$ is a unital finite index inclusion, then $D \prec_M B$.*

Using the previous two lemmas we can prove the following well known result for which we could not find a reference.

Lemma 1.10.4. *Let $A, B \subset (M, \tau)$ be, possibly nonunital, embeddings. Let $n \in \mathbb{Z}$ be a nonzero positive integer and let $p \in A^n$ and $q \in B^n$ be nonzero projections. If $pA^n p \prec_{M^n} qB^n q$, then $A \prec_M B$.*

Proof. Assume that $pA^n p \prec_{M^n} qB^n q$. By Lemma 1.10.2, this implies that $A^n \prec_{M^n} B^n$. Then, using Lemma 1.10.3, we have that $1 \otimes A \prec_{M^n} 1 \otimes B$. In other words, there exists a nonzero A-B-subbimodule $\mathcal{K}$ of $(\mathbb{C}^{n^2} \otimes 1_A L^2(M) 1_B)$ that is finitely generated as a right B-module. By projecting $\mathcal{K}$ onto an appropriate copy of $1_A L^2(M) 1_B$, we find a nonzero A-B-subbimodule $\mathcal{H}$ of $1_A L^2(M) 1_B$ that is finitely generated as a right B-module. This ends the proof. $\square$

We also need a stronger notion than intertwining-by-bimodules called *full embedding*.

Definition 1.10.5. Let (M, τ) be a tracial von Neumann algebra and let $A, B \subset M$ be possibly nonunital von Neumann subalgebras. Denote the unit of A by 1_A. We say that A *embeds fully into* B *inside* M, and we write $A \prec^f_M B$, if $Ap \prec_M B$ for every nonzero projection $p \in 1_A M 1_A \cap A'$.

One of the advantages that full embedding has over intertwining is that the relation '$\prec^f_M$' is transitive, while the relation '$\prec_M$' need not be. Let us go into this a bit deeper.

A von Neumann algebra M is called *diffuse* if it has no minimal projections. Using the third characterisation of 1.10.1, we have that $M \prec_M \mathbb{C}$ if and only if M is not diffuse. With this in mind, we can see that the relation '$\prec_M$' need not be transitive. Indeed, let p be a nontrivial projection in a diffuse tracial von Neumann algebra (M, τ), then $M \prec_M pMp + \mathbb{C}(1-p)$ and $pMp + \mathbb{C}(1-p) \prec_M \mathbb{C}$, but $M \not\prec_M \mathbb{C}$. On the other hand, Lemma 1.10.6 below shows that the relation '$\prec^f_M$' is always transitive. A proof of it can be found in [Va08, Lemma 3.7].

Lemma 1.10.6. *Let (M, τ) be a tracial von Neumann algebra and let $A, B, D \subset (M, \tau)$ be possibly nonunital embeddings. If $A \prec_M B$ and $B \prec^f_M D$, then $A \prec_M D$.*

A way to obtain full embedding is by quasi-regularity. We first recall the definition of quasi-regularity.

Definition 1.10.7. Let (M, τ) be a tracial von Neumann algebra and $N \subset M$ a von Neumann subalgebra. We denote by $\mathrm{QN}_M(N)$ the *quasi-normalizer* of N inside M, i.e. the unital $*$-algebra defined by

$$\left\{ a \in M \;\middle|\; \exists b_1, \ldots, b_k \in M, \exists d_1, \ldots, d_r \in M :\ Na \subset \sum_{i=1}^{k} b_i N,\ aN \subset \sum_{j=1}^{r} N d_j \right\}.$$

We call $N \subset M$ *quasi-regular* if $\mathrm{QN}_M(N)'' = M$.

In the following subsection we introduce the notion of almost normal subgroups. It will turn out that $L(\Lambda) \subset L(\Gamma)$ is quasi-regular whenever $\Lambda \leq \Gamma$ is an almost normal subgroup, see Lemma 1.11.4. Most of the quasi-regular inclusions we encounter will be of this form.

We say that $N \subset M$ is an *irreducible* inclusion of von Neumann algebras if the relative commutant is trivial, i.e. when $N' \cap M = \mathbb{C}1$. When M comes equipped with a normal faithful tracial state τ, this is equivalent with saying that $L^2(M)$ is an irreducible N-M-bimodule. Let us explain this. So assume that (M, τ) is a tracial von Neumann algebra and $N \subset M$ is a von Neumann subalgebra. Whenever $\mathcal{H}$ is a Hilbert subspace of $L^2(M)$, we write $p_{\mathcal{H}}$ for the orthogonal projection of $L^2(M)$ onto $\mathcal{H}$. Now note that the map $\mathcal{H} \to p_{\mathcal{H}}$ is a bijection between the N-M-subbimodules of $L^2(M)$ and the projections of $N' \cap M$. In this way, we indeed see that $N \subset (M, \tau)$ is an irreducible inclusion if and only if $L^2(M)$ is an irreducible N-M-bimodule.

The following three results are well known to experts. Again we could not find any reference, but give a proof ourselves.

Lemma 1.10.8. *Let (M, τ) be a tracial von Neumann algebra and let $A, B \subset M$ be possibly nonunital von Neumann subalgebras. If $A \prec_M B$ and $\mathrm{QN}_{1_A M 1_A}(A)''$ is an irreducible von Neumann subalgebra of $1_A M 1_A$, then $A \prec_M^f B$.*

Proof. Let $p \in A' \cap 1_A M 1_A$ be a nonzero projection. We prove that there exists a nonzero Ap-B-subbimodule $\mathcal{K}$ of $pL^2(M)1_B$ with finite B-dimension.

Since $A \prec_M B$, there exists a nonzero A-B-subbimodule $\mathcal{H}$ of $1_A L^2(M) 1_B$ with $\dim_{-B}(\mathcal{H}) < \infty$. Since $\mathrm{QN}_{1_A M 1_A}(A)''$ is an irreducible von Neumann subalgebra of $1_A M 1_A$, we have that $1_A L^2(M) 1_B$ is an irreducible $\mathrm{QN}_{1_A M 1_A}(A)''$-$(1_B M 1_B)$-bimodule. This then implies that $1_A L^2(M) 1_B = \overline{\mathrm{span}}(\mathrm{QN}_{1_A M 1_A}(A)'' \mathcal{H} M 1_B)$. Therefore

$$p\,\mathrm{QN}_{1_A M 1_A}(A)'' \mathcal{H} M 1_B \neq \{0\},$$

or equivalently $p\,\mathrm{QN}_{1_A M 1_A}(A)'' \mathcal{H} \neq \{0\}$. Hence, there exists an element $v \in \mathrm{QN}_{1_A M 1_A}(A)$ such that $pv\mathcal{H}$ is nonzero. Write $\mathcal{K} := \overline{\mathrm{span}}(pAv\mathcal{H})$, then $\mathcal{K}$ is a nonzero Ap-B-subbimodule of $pL^2(M)1_B$ with finite B-dimension. $\square$

Let (M, τ) be a tracial von Neumann algebra and let $A, B \subset (M, \tau)$ be von Neumann subalgebras. If $A \prec_M B$ and $B \prec_M A$, then $L^2(M)$ does admit both a nonzero A-B-subbimodule with a finite B-dimension and one with a finite A-dimension. However this does not immediately imply that $L^2(M)$ admits a nonzero finite index A-B-subbimodule, but we do have the following proposition.

Proposition 1.10.9. *Let (M, τ) be a tracial von Neumann algebra and let $A, B \subset (M, \tau)$ be possibly nonunital von Neumann subalgebras. If $A \prec_M^f B$, $B \prec_M A$ and B is quasi-regular inside $1_B M 1_B$, then there exists a nonzero finite index A-B-subbimodule of $1_A L^2(M) 1_B$.*

Proof. Since $B \prec_M A$, there exists a nonzero subbimodule ${}_A\mathcal{H}_B$ of ${}_A(1_A L^2(M)1_B)_B$ with finite A-dimension. Consider the A-1_BM1_B-subbimodule $\overline{\text{span}}(\mathcal{H}M1_B)$ of $L^2(M)1_B$. There exists a projection $p \in A' \cap 1_AM1_A$ such that $\overline{\text{span}}(\mathcal{H}M1_B) = pL^2(M)1_B$. Since B is quasi-regular inside 1_BM1_B, we therefore get that $pL^2(M)1_B$ is densely spanned by the A-B-subbimodules

$$ {}_A\overline{\text{span}}(\mathcal{H}vB)_B \text{ with } v \in \mathrm{QN}_{1_BM1_B}(B). $$

Now since $A \prec^f_M B$, there also exists a nonzero A-B-subbimodule $\mathcal{K}$ of $pL^2(M)1_B$ with finite B-dimension. Take an element $v \in \mathrm{QN}_{1_BM1_B}(B)$ such that the orthogonal projection p_v of $pL^2(M)1_B$ onto $\overline{\text{span}}(\mathcal{H}vB)$ satisfies $p_v(\mathcal{K}) \neq \{0\}$. Then ${}_A(p_v(\mathcal{K}))_B$ is a nonzero subbimodule of ${}_A\overline{\text{span}}(\mathcal{H}vB)_B$ with $\dim_{-B}(p_v(\mathcal{K})) \leq \dim_{-B}(\mathcal{K}) < \infty$. Since $\dim_{A-}(\overline{\text{span}}(\mathcal{H}vB)) < \infty$, also $\dim_{A-}(p_v(\mathcal{K})) < \infty$. This ends the proof. $\square$

We end this section with the following lemma.

Lemma 1.10.10. *Let (M,τ) be a tracial von Neumann algebra, let $A \subset M$ be a possibly nonunital von Neumann subalgebra and let $B \subset M$ be a unital von Neumann subalgebra. Denote the unit of A by 1_A.*

- *If $p \in A' \cap 1_AM1_A$ is a nonzero projection and $Ap \prec_M B$, then there exists a nonzero projection $p_0 \in A' \cap 1_AM1_A$ such that $p_0 \leq p$ and Ap_0 is amenable relative to B inside M.*
- *If $A \prec^f_M B$, then A is amenable relative to B inside M.*

Proof. To prove the first statement, we assume that $Ap \prec_M B$ for some nonzero projection $p \in A' \cap 1_AM1_A$. By definition, we have that there exists a nonzero projection $q \in B^n$, a normal unital $*$-homomorphism $\psi : Ap \to qB^nq$ and a nonzero partial isometry $v \in (M_{1,n}(\mathbb{C}) \otimes pM)q$ such that $av = v\psi(a)$ for all $a \in Ap$. Put $p_0 = vv^* \in (Ap)' \cap pMp$ and

$$ \varphi : p_0\langle M, e_B\rangle p_0 \to \mathbb{C} : T \mapsto \frac{(\mathrm{Tr}_{M_n(\mathbb{C})} \otimes \tau_M)(q)}{\tau_M(p_0)} \mathrm{Tr}(e_{qB^nq} v^* T v e_{qB^nq}), $$

where Tr denotes the canonical semi-finite trace on $\langle qM^nq, e_{qB^nq}\rangle$. Then φ is a positive linear functional on $p_0\langle M, e_B\rangle p_0$.

Now let $T \in p_0\langle M, e_B\rangle p_0$ and $x \in Ap_0$. Then,

$$ \begin{aligned} \mathrm{Tr}(e_{qB^nq}v^*xTve_{qB^nq}) &= \mathrm{Tr}(e_{qB^nq}\psi(x)v^*Tve_{qB^nq}) = \mathrm{Tr}(\psi(x)e_{qB^nq}v^*Tve_{qB^nq}) \\ &= \mathrm{Tr}(e_{qB^nq}v^*Tve_{qB^nq}\psi(x)) = \mathrm{Tr}(e_{qB^nq}v^*Tv\psi(x)e_{qB^nq}) \\ &= \mathrm{Tr}(e_{qB^nq}v^*Txve_{qB^nq}). \end{aligned} $$

Hence φ is Ap_0-central. Furthermore, for every $x \in p_0 M p_0$, we have that

$$\begin{aligned}\varphi(x) &= \frac{(\mathrm{Tr}_{M_n(\mathbb{C})} \otimes \tau_M)(q)}{\tau_M(p_0)} \mathrm{Tr}(e_{qB^n q} v^* x v e_{qB^n q}) \\ &= \frac{(\mathrm{Tr}_{M_n(\mathbb{C})} \otimes \tau_M)(q)}{\tau_M(p_0)} \mathrm{Tr}(E_{qB^n q}(v^* x v) e_{qB^n q}) \\ &= (\mathrm{Tr}_{M_n(\mathbb{C})} \otimes \tau_M)(v^* x v)/\tau_M(p_0) \\ &= \tau_M(v v^* x)/\tau_M(p_0) \\ &= \tau_{p_0 M p_0}(x).\end{aligned}$$

This ends the proof of the first statement.

To prove the second statement, we assume that $A \prec^f_M B$. Using what we just found together with a standard maximality argument, we have that there exist nonzero projections $p_i \in A' \cap 1_A M 1_A$ such that $\sum_{i \in \mathbb{N}} p_i = 1_A$ and Ap_i is amenable relative to B inside M. Thus, for every $i \in \mathbb{N}$, we have an Ap_i-central state φ_i on $p_i \langle M, e_B \rangle p_i$ such that $\varphi_i|_{p_i M p_i} = \tau_{p_i M p_i}$. Define

$$\varphi : 1_A \langle M, e_B \rangle 1_A \to \mathbb{C} : x \mapsto \sum_i \varphi_i(p_i x p_i) \tau_{1_A M 1_A}(p_i).$$

Then φ is an A-central state on $1_A \langle M, e_B \rangle 1_A$ such that $\varphi|_{1_A M 1_A} = \tau_{1_A M 1_A}$. In other words, A is amenable relative to B inside M. □

1.11 Baumslag-Solitar groups and HNN extensions

For all $n, m \in \mathbb{Z} \setminus \{0\}$, the Baumslag-Solitar group $\mathrm{BS}(n, m)$ is defined as the group generated by a and b subject to the relation $ba^n b^{-1} = a^m$. So,

$$\mathrm{BS}(n, m) := \langle a, b \mid ba^n b^{-1} = a^m \rangle .$$

The Baumslag-Solitar groups were introduced in [BS62] as the first examples of two generator non-Hopfian groups with a single defining relation. Ever since, they have been playing an important role in many different areas of mathematics. The following are two examples of this.

- In theoretical informatics they were the first groups that were known to be asynchronous automatic but not automatic (see e.g. [ECH+92]);

- In geometric group theory they provide easy examples of groups that are not isomorphic to a subgroup of a hyperbolic group (see e.g. [GS91]).

Since they play such important roles in mathematics, It is a natural problem to classify von Neumann algebras arising in some way from Baumslag-Solitar groups.

The following facts will be useful later on. Whenever $|n| = 1$ or $|m| = 1$, the normal closure of $\langle a \rangle$ is an abelian normal subgroup of $\mathrm{BS}(n,m)$ such that the quotient is infinite cyclic. So in that case, the group $\mathrm{BS}(n,m)$ is solvable, hence amenable. Whenever $|n| \geq 2$ and $|m| \geq 2$, the subgroup $\langle b, aba^{-1} \rangle \leq \mathrm{BS}(n,m)$ is, by Lemma 1.11.1 below, isomorphic with the free group $\mathbb{F}_2$. So in that case, $\mathrm{BS}(n,m)$ is nonamenable. In [Mo91], the Baumslag-Solitar groups were classified up to isomorphism: $\mathrm{BS}(n,m) \cong \mathrm{BS}(p,q)$ if and only if $\{n,m\} = \{\varepsilon p, \varepsilon q\}$ for some $\varepsilon \in \{-1,1\}$. So the $\mathrm{BS}(n,m)$ with $2 \leq n \leq |m|$ form a complete list of all nonamenable Baumslag-Solitar groups up to isomorphism. Finally by [St05, Exemple 2.4], the group $\mathrm{BS}(n,m)$ is icc if and only if $|n| \neq |m|$.

Now, let us introduce the notion of HNN extension of groups ([HNN49]). Let G be a group, $H < G$ a subgroup and $\theta : H \to G$ an injective group homomorphism. The *HNN extension* $\mathrm{HNN}(G,H,\theta)$ is defined by the presentation

$$\mathrm{HNN}(G,H,\theta) = \langle G, b \mid \theta(h) = bhb^{-1} \ \text{ for all } \ h \in H \rangle \ .$$

Elements of $\mathrm{HNN}(G,H,\theta)$ can be expressed in a 'reduced' way using as letters the elements of G and the letters $b^{\pm 1}$. More precisely, we have the following lemma.

Lemma 1.11.1 (Britton's lemma, [Br63]). *Consider the expression* $g = g_0 b^{n_1} g_1 b^{n_2} \cdots b^{n_k} g_k$ *with* $k \geq 0$, $g_0, g_k \in G$, $g_1, \ldots, g_{k-1} \in G \setminus \{e\}$ *and* $n_1, \ldots, n_k \in \mathbb{Z} \setminus \{0\}$. *We call this expression reduced if the following two conditions hold:*

- *for every* $i \in \{1, \ldots, k-1\}$ *with* $n_i > 0$ *and* $n_{i+1} < 0$, *we have* $g_i \notin H$,
- *for every* $i \in \{1, \ldots, k-1\}$ *with* $n_i < 0$ *and* $n_{i+1} > 0$, *we have* $g_i \notin \theta(H)$.

If the above expression for g *is reduced, then* $g \neq e$ *in the group* $\mathrm{HNN}(G,H,\theta)$, *unless* $k = 0$ *and* $g_0 = e$. *In particular, the natural homomorphism of* G *to* $\mathrm{HNN}(G,H,\theta)$ *is injective.*

The number $\sum_{i=1}^{k} |n_i|$ appearing in a reduced expression of g is called the *b-length* of g. Observe that it does not depend on the choice of the reduced expression.

Note that the Baumslag-Solitar groups are one of the easiest examples of HNN extensions. Indeed, $\mathrm{BS}(n,m) = \mathrm{HNN}(\mathbb{Z}, n\mathbb{Z}, \theta)$, where $\theta(n) = m$. In this way, we can use Lemma 1.11.1 whenever we are dealing with Baumslag-Solitar groups. More precisely, consider the expression $g = a^{l_0} b^{n_1} a^{l_1} b^{n_2} \cdots b^{n_k} a^{l_k}$ with $k \geq 0$, $l_0, l_k \in \mathbb{Z}$, $l_1, \ldots, l_{k-1} \in \mathbb{Z} \setminus \{0\}$ and $n_1, \ldots, n_k \in \mathbb{Z} \setminus \{0\}$. Then this expression is reduced if the following two conditions hold:

- for every $i \in \{1, \ldots, k-1\}$ with $n_i > 0$ and $n_{i+1} < 0$, we have $n \not| \ l_i$;
- for every $i \in \{1, \ldots, k-1\}$ with $n_i < 0$ and $n_{i+1} > 0$, we have $m \not| \ l_i$.

Lemma 1.11.2. *Let $n, m \in \mathbb{Z}$ satisfy $2 \leq n \leq |m|$. The centralizer $\mathcal{C}$ of $\langle a^n \rangle$ inside $\mathrm{BS}(n,m)$ is nonamenable.*

Proof. We have the following three cases to consider.

Case 1. ($|m| \neq 2$). Define $G := \langle a^{\mathbb{Z}}, b^{-1} a^{\mathbb{Z}} b \rangle \leq \mathrm{BS}(n,m)$. We first show that G is an amalgamated free product of two copies of $\mathbb{Z}$ over a copy of $\mathbb{Z}$ embedded as $n\mathbb{Z}$ and $m\mathbb{Z}$ respectively. Write $H := \{c, d \mid c^n = d^m\}$ and define the homomorphism $\alpha : H \to G$ by $\alpha(c) = a$ and $\alpha(d) = b^{-1}ab$. Note that α is well defined and surjective. We show that α is also injective. To that end we fix $g \in \mathrm{Ker}(\alpha)$. Write $g = c^{n_0} d^{m_1} c^{n_1} \ldots d^{m_k} c^{n_k}$ with $k \geq 0$, $n_0 \in \mathbb{Z}$, $n_1, \ldots, n_k \in \mathbb{Z} \setminus n\mathbb{Z}$ and $m_1, \ldots, m_k \in \mathbb{Z} \setminus m\mathbb{Z}$. Then,

$$\begin{aligned} e = \alpha(g) &= \alpha(c^{n_0} d^{m_1} c^{n_1} \ldots d^{m_k} c^{n_k}) \\ &= \alpha(c)^{n_0} \alpha(d)^{m_1} \alpha(c)^{n_1} \ldots \alpha(d)^{m_k} \alpha(c)^{n_k} \\ &= a^{n_0} b^{-1} a^{m_1} b a^{n_1} \ldots b^{-1} a^{m_k} b a^{n_k}. \end{aligned}$$

Note that this last expression is reduced inside $\mathrm{BS}(n,m)$. So, by Lemma 1.11.1, we have that $k = 0$ and $n_0 = 0$. But then $g = e$ and hence $\mathrm{Ker}(\alpha) = \{e\}$. Altogether, we see that G is indeed an amalgamated free product of two copies of $\mathbb{Z}$ over a copy of $\mathbb{Z}$ embedded as $n\mathbb{Z}$ and $m\mathbb{Z}$ respectively. In particular G is nonamenable by the remark following Proposition 23 of [dlHP11]. Since G is a subgroup of $\mathcal{C}$, we have that $\mathcal{C}$ is also nonamenable.

Case 2. ($m = 2$). So $n = m = 2$. In this case $\mathcal{C}$ and $\mathrm{BS}(n,m)$ coincide. In particular, $\mathcal{C}$ is nonamenable.

Case 3. ($m = -2$). So $n = 2$ and $m = -2$. Let $g \in \mathrm{BS}(n,m) \setminus \mathcal{C}$. Then $a^2 g = g a^{-2}$, and hence $a^2 g b = g a^{-2} b = g b a^2$. This shows that $\mathrm{BS}(n,m) = \mathcal{C} \sqcup \mathcal{C} b^{-1}$. In other words, $\mathcal{C}$ is an index 2 normal subgroup of $\mathrm{BS}(n,m)$. In particular, $\mathcal{C}$ is nonamenable. □

Since $\mathrm{BS}(n,m)$ is an HNN extension, we have that it acts on its *Bass-Serre tree*. Recall from [Se80] that the Bass-Serre tree T is defined as follows:

$$V(T) = \mathrm{BS}(n,m)/\langle a\rangle \text{ and } E^+(T) = \mathrm{BS}(n,m)/\langle a^n\rangle,$$

where $V(T)$ denotes the set of vertices of T and $E^+(T)$ denotes the set of positive oriented edges of T. The *source* map $s : E^+(T) \to V(T)$ and the *range* map $r : E^+(T) \to V(T)$ are defined by

$$s(g\langle a^n\rangle) = g\langle a\rangle \text{ and } r(g\langle a^n\rangle) = gb^{-1}\langle a\rangle \text{ for every } g \in \mathrm{BS}(n,m).$$

The group $\mathrm{BS}(m,n)$ acts on T by left multiplication.

In general, when Γ is a group acting on a tree T, we call a group element *elliptic* if it admits a fixed point, otherwise we call it *hyperbolic*. The following lemma is well known and follows immediately from [Se80, Proposition 25 and Proposition 26].

Lemma 1.11.3. *Let Γ be a group acting on a tree T.*

1. *If $g \in \Gamma$ is a hyperbolic element, then g^z is hyperbolic for every nonzero integer z.*
2. *If $g, h \in \Gamma$ are elliptic elements such that gh is elliptic, then g and h have a common fixed point.*

Now we introduce the notion of almost normal subgroup. Let Γ be group and $\Lambda < \Gamma$ a subgroup. Define the following functions on Γ having values in $\mathbb{N} \cup \{\infty\}$:

1. $r(g) = [\Lambda : \Lambda \cap g\Lambda g^{-1}] =$ the number of right Λ-cosets in the double coset $\Lambda g\Lambda$;
2. $l(g) = r(g^{-1}) = [\Lambda : \Lambda \cap g^{-1}\Lambda g] =$ the number of left Λ-cosets in the double coset $\Lambda g\Lambda$.

If $l(g)$ is finite for every $g \in \Gamma$, we say that Λ is an *almost normal* subgroup of Γ. In that case, we also call (Γ, Λ) a *Hecke pair*.

Remark. *In the literature, e.g. [Tz03], the function r is usually denoted by L and the function l is usually denoted by R. Let us justify our choice of notation. Let (P, τ) be a tracial von Neumann algebra and let $\mathrm{BS}(n,m) \curvearrowright P$ be a trace preserving action. Set $N := P \rtimes \langle a\rangle$ and define the N-N-bimodule $\mathcal{K}_g := \overline{\mathrm{span}}\, Nu_gN$ for every $g \in \mathrm{BS}(n,m)$. Then the left dimension $\dim_{N-}(\mathcal{K}_g)$ equals $l(g)$ and the right dimension $\dim_{-N}(\mathcal{K}_g)$ equals $r(g)$.*

In the previous subsection, we came across the notion of quasi-regularity. The following well known result provides many examples of quasi-regular inclusions of von Neumann algebras. The result is for instance mentioned in Subsection 2.1 of [Fa09].

Lemma 1.11.4. *Let Γ be a countable discrete group. If Λ is an almost normal subgroup of Γ, then $L(\Lambda)$ is a quasi-regular von Neumann subalgebra of $L(\Gamma)$.*

Proof. Let Γ be a countable discrete group and assume that $\Lambda \leq \Gamma$ is an almost normal subgroup. Write $M := L(\Gamma)$ and $N := L(\Lambda)$. Fix $g \in \Gamma$. We show that $u_g \in \mathrm{QN}_M(N)$.

Since Λ is an almost normal subgroup of Γ, we have that $g\Lambda$ is contained in a finite union of right cosets. Say $g\Lambda \subset \bigsqcup_{i=1}^n \Lambda h_i$ where $h_1, \ldots, h_n \in \Gamma$. Then we have that $u_g N \subset \sum_{i=1}^n N u_{h_i}$. Completely analogous, $N u_g \subset \sum_{j=1}^m u_{k_j} N$ for some $k_1, \ldots, k_m \in \Gamma$. Altogether this shows that $u_g \in \mathrm{QN}_M(N)$.

So far we have found that $\{u_g \mid g \in \Gamma\}$ is a subset of $\mathrm{QN}_M(N)$. Since $\{u_g \mid g \in \Gamma\}$ generates the whole of M as a von Neumann algebra, this ends the proof. $\square$

Note that $(\mathrm{BS}(n,m), \langle a \rangle)$ is a Hecke pair. When considering this pair, $l(g)$ is the smallest nonzero positive integer such that $ga^{l(g)} \in \langle a \rangle g$. Similarly, $r(g)$ is the smallest nonzero positive integer such that $a^{r(g)} g \in g\langle a \rangle$. Writing $k = \gcd(|n|, |m|)$, $n_0 = n/k$, $m_0 = m/k$ and

$$\mathcal{F} := \{k|n_0|^s |m_0|^t \mid s, t \in \mathbb{N} \text{ with } s + t > 0\},$$

we have that $\mathcal{F} \cup \{1\} = \{l(g) \mid g \in \mathrm{BS}(n,m)\}$.

We end this section with the notion of quasi-centralizer. The *quasi-centralizer* $\mathrm{QC}_\Gamma(\Lambda)$ of an inclusion $\Lambda \leq \Gamma$ of groups is defined as

$$\mathrm{QC}_\Gamma(\Lambda) := \bigcup_{\substack{\Lambda_1 \leq \Lambda \\ \text{finite index}}} \mathrm{C}_\Gamma(\Lambda_1),$$

where $\mathrm{C}_\Gamma(\Lambda_1)$ denotes the centralizer of $\Lambda_1 \leq \Gamma$. Note that $\mathrm{QC}_\Gamma(\Lambda)$ is a normal subgroup of Γ.

Lemma 1.11.5. *Let $n, m \in \mathbb{Z}$ be nonzero integers. Then,*

$$\mathrm{QC}_{\mathrm{BS}(n,m)}(\langle a \rangle) = \{g \in \mathrm{BS}(n,m) \mid g a^{l(g)} g^{-1} = a^{l(g)}\}.$$

Proof. It suffices to show that

$$\mathrm{QC}_{\mathrm{BS}(n,m)}(\langle a \rangle) \subset \{g \in \mathrm{BS}(n,m) \mid g a^{l(g)} g^{-1} = a^{l(g)}\},$$

since the converse inclusion is obvious. So fix an element g of the quasi-centralizer of $\langle a\rangle \subset \mathrm{BS}(n,m)$. Then there exists a nonzero positive integer l such that $ga^lg^{-1} = a^l$. This implies that $a^l \in \langle a\rangle \cap g^{-1}\langle a\rangle g = \langle a^{l(g)}\rangle$ or in other words that $l(g)$ divides l. Writing, $l = l_0 l(g)$ we have that

$$a^l = ga^lg^{-1} = (ga^{l(g)}g^{-1})^{l_0}.$$

Now $ga^{l(g)}g^{-1} = a^r$, for some $r \in \{r(g), -r(g)\}$. So altogether $a^l = a^{rl_0}$ and therefore $r = l/l_0 = l(g)$. We conclude that $ga^{l(g)}g^{-1} = a^r = a^{l(g)}$. $\square$

1.12 HNN extensions of von Neumann algebras

In [Ue04], HNN extensions of general von Neumann algebras were first introduced. For the case of tracial von Neumann algebras, an easier approach to the construction was given by Fima and Vaes in [FV12]. In this section, we recall this approach.

Let Γ be a countable discrete group, $\Lambda < \Gamma$ a subgroup and $\theta : \Lambda \to \Gamma$ an injective group homomorphism. Then $M := L(\Gamma)$ is a tracial von Neumann algebra, $N := L(\Lambda)$ is a von Neumann subalgebra of M and $\theta : N \to M : u_\lambda \mapsto u_{\theta(\lambda)}$ is a trace preserving embedding of N into M. In [FV12], a tracial von Neumann algebra $\mathrm{HNN}(M,N,\theta)$ is associated to every triplet (M,N,θ) where M is a tracial von Neumann algebra, N is a von Neumann subalgebra of M and θ is a trace preserving embedding of N into M. The construction satisfies $L(\mathrm{HNN}(\Gamma,\Lambda,\theta)) = \mathrm{HNN}(L(\Gamma), L(\Lambda), \theta)$.

So fix a triplet (M,N,θ) where M is a tracial von Neumann algebra, N is a von Neumann subalgebra of M and θ is a trace preserving embedding of N into M. For $\varepsilon \in \{-1,1\}$ define

$$N_\varepsilon = \begin{cases} N & \text{if } \varepsilon = 1, \\ \theta(N) & \text{if } \varepsilon = -1. \end{cases}$$

Define $\theta^\varepsilon : N_\varepsilon \to N_{-\varepsilon}$ in the obvious way.

For $n \geq 1$ and $\varepsilon_1, \dots, \varepsilon_n \in \{-1,1\}$ define the M-M-bimodule

$$H_{\varepsilon_1,\dots,\varepsilon_n} = K_0 \otimes_N \dots \otimes_N K_n,$$

where $K_0 = K_n = L^2(M)$ and, for $1 \leq i \leq n-1$,

$$K_i = \begin{cases} L^2(M) & \text{if } \varepsilon_i = \varepsilon_{i+1}, \\ L^2(M) \ominus L^2(N_{\varepsilon_i}) & \text{if } \varepsilon_i \neq \varepsilon_{i+1}. \end{cases}$$

We view $K_0 = L^2(M)$ as an M-N-bimodule, where the left M-action is the usual one and the right N-action is given by $\xi \cdot x = \xi x$ if $\varepsilon_1 = -1$ and $\xi \cdot x = \xi\theta(x)$ if $\varepsilon_1 = 1$. Similarly, we view $K_n = L^2(M)$ as an N-M-bimodule, where the right M-action is the usual one and the left N action is given by $x \cdot \xi = x\xi$ if $\varepsilon_n = 1$ and $x \cdot \xi = \theta(x)\xi$ if $\varepsilon_n = -1$. Finally, for $1 \leq i \leq n-1$, we view $\mathcal{K}_i$ as an N-N-bimodule in the following way.

- The left N-action is given by $x \cdot \xi = \begin{cases} x\xi & \text{if } \varepsilon_i = 1, \\ \theta(x)\xi & \text{if } \varepsilon_i = -1. \end{cases}$
- The right N-action is given by $\xi \cdot x = \begin{cases} \xi\theta(x) & \text{if } \varepsilon_{i+1} = 1, \\ \xi x & \text{if } \varepsilon_{i+1} = -1. \end{cases}$

Define the M-M-bimodule

$$H = L^2(M) \oplus \bigoplus_{n\geq 1, \varepsilon_1,\ldots,\varepsilon_n \in \{-1,1\}} H_{\varepsilon_1,\ldots,\varepsilon_n}.$$

Every element $x \in M$ can be viewed as a vector in $L^2(M)$ that we denote by $\hat{x}$. Now let $\varepsilon \in \{-1, 1\}$. We define a unitary $u^\varepsilon \in B(H)$ in the following way.

- If $\xi \in L^2(M)$ we define $u^\varepsilon \xi = \hat{1} \otimes \xi \in H_\varepsilon$.
- If $\xi \in H_{\varepsilon_1,\ldots,\varepsilon_n}$ with $n \geq 1$ and $\varepsilon_1 = \varepsilon$ we define $u^\varepsilon \xi = \hat{1} \otimes \xi \in H_{\varepsilon,\varepsilon_1,\ldots,\varepsilon_n}$.
- If $\xi = \hat{x} \otimes \xi_0 \in H_{\varepsilon_1,\ldots,\varepsilon_n}$ with $n \geq 1$, $\varepsilon_1 \neq \varepsilon$ and $x \in M$, $\xi_0 \in H_{\varepsilon_2,\ldots,\varepsilon_n}$ we define

$$u^\varepsilon(\hat{x} \otimes \xi_0) = \begin{cases} \hat{1} \otimes \hat{x} \otimes \xi_0 \in H_{\varepsilon,\varepsilon_1,\ldots,\varepsilon_n} & \text{if } x \in M \ominus N_\varepsilon, \\ \theta^\varepsilon(x)\xi_0 \in H_{\varepsilon_2,\ldots,\varepsilon_n} & \text{if } x \in N_\varepsilon. \end{cases}$$

In order to see that u^ε indeed extends to a unitary on H, we first show that u^ε extends to an isometry on H. For that, fix

$$\xi, \eta \in L^2(M) \cup \left(\bigcup_{n\geq 1, \varepsilon_1,\ldots,\varepsilon_n \in \{-1,1\}} H_{\varepsilon_1,\ldots,\varepsilon_n} \right).$$

We need to prove that $\langle \xi, \eta \rangle = \langle u^\varepsilon \xi, u^\varepsilon \eta \rangle$. There are many cases to be checked, but only the following three (up to symmetry) are nontrivial.

- **Case 1.** $\xi \in L^2(M)$ and $\eta = \hat{x} \otimes \eta_0 \in \mathcal{H}_{-\varepsilon,\varepsilon}$ with $x \in N_\varepsilon$ and $\eta_0 \in H_\varepsilon$;

- **Case 2.** $\xi \in H_{\varepsilon_1,\ldots,\varepsilon_n}$ and $\eta = \hat{x} \otimes \eta_0 \in H_{-\varepsilon,\varepsilon,\varepsilon_1,\ldots,\varepsilon_n}$ with $n \geq 1$, $\varepsilon_1 = \varepsilon$ and $x \in N_\varepsilon$, $\eta_0 \in H_{\varepsilon,\varepsilon_1,\ldots,\varepsilon_n}$;
- **Case 3.** $\xi = \hat{x} \otimes \xi_0 \in H_{\varepsilon_1,\ldots,\varepsilon_n}$ and $\eta = \hat{y} \otimes \eta_0 \in H_{\varepsilon_1,\ldots,\varepsilon_n}$ with $n \geq 1$, $\varepsilon_1 \neq \varepsilon$ and $x, y \in N_\varepsilon$, $\xi_0, \eta_0 \in H_{\varepsilon_2,\ldots,\varepsilon_n}$.

In the first case we have that

$$\langle u^\varepsilon \xi, u^\varepsilon \eta \rangle = \langle u^\varepsilon \xi, u^\varepsilon(\hat{x} \otimes \eta_0) \rangle = \langle 1 \otimes \xi, \theta^\varepsilon(x)\eta_0 \rangle.$$

Since η_0 is an element of $(L^2(M) \ominus L^2(N_{-\varepsilon})) \otimes_N L^2(M)$ inside H_ε, we have that $\theta^\varepsilon(x)\eta_0$ is also an element of $(L^2(M) \ominus L^2(N_{-\varepsilon})) \otimes_N L^2(M)$ inside H_ε. Now $(L^2(M) \ominus L^2(N_{-\varepsilon})) \otimes_N L^2(M)$ is orthogonal to $L^2(N_{-\varepsilon}) \otimes_N L^2(M)$ inside H_ε. Therefore $\langle 1 \otimes \xi, \theta^\varepsilon(x)\eta_0 \rangle = 0$ and hence $\langle u^\varepsilon \xi, u^\varepsilon \eta \rangle = 0 = \langle \xi, \eta \rangle$.

The second case follows by exactly the same arguments as the first case.

Finally, the third case follows immediately from the following calculation:

$$\begin{aligned}
\langle u^\varepsilon \xi, u^\varepsilon \eta \rangle &= \langle u^\varepsilon(\hat{x} \otimes \xi_0), u^\varepsilon(\hat{y} \otimes \eta_0) \rangle = \langle \theta^\varepsilon(x)\xi_0, \theta^\varepsilon(y)\eta_0 \rangle \\
&= \langle \hat{1} \otimes \theta^\varepsilon(x)\xi_0, \hat{1} \otimes \theta^\varepsilon(y)\eta_0 \rangle = \langle \hat{1} \otimes (x \cdot \xi_0), \hat{1} \otimes (y \cdot \eta_0) \rangle \\
&= \langle (\hat{1} \cdot x) \otimes \xi_0, (\hat{1} \cdot y) \otimes \eta_0 \rangle = \langle \hat{x} \otimes \xi_0, \hat{y} \otimes \eta_0 \rangle \\
&= \langle \xi, \eta \rangle.
\end{aligned}$$

Altogether, we have showed that u^ε extends to an isometry on H. Let us now show that $u^\varepsilon x u^{-\varepsilon} = \theta^\varepsilon(x)$ for every $x \in N_\varepsilon$. For that, fix $x \in N_\varepsilon$ and

$$\xi \in L^2(M) \cup \left(\bigcup_{n \geq 1, \varepsilon_1,\ldots,\varepsilon_n \in \{-1,1\}} H_{\varepsilon_1,\ldots,\varepsilon_n} \right).$$

We need to prove that $(u^\varepsilon x u^{-\varepsilon})(\xi) = \theta^\varepsilon(x)\xi$. Again there are many cases to be checked, but only the following three are nontrivial.

- **Case 1.** $\xi = \hat{y} \otimes \xi_0 \in H_\varepsilon$ with $y \in N_{-\varepsilon}$ and $\xi_0 \in L^2(M)$;
- **Case 2.** $\xi = \hat{y} \otimes \xi_0 \in H_{\varepsilon,\varepsilon,\varepsilon_3,\ldots,\varepsilon_n}$ with $n \geq 2$, $y \in N_{-\varepsilon}$, $\xi_0 \in H_{\varepsilon,\varepsilon_3,\ldots,\varepsilon_n}$;
- **Case 3.** $\xi = \hat{y} \otimes \xi_0 \in H_{\varepsilon,-\varepsilon,\varepsilon_3,\ldots,\varepsilon_n}$ with $n \geq 2$, $y \in N_{-\varepsilon}$, $\xi_0 \in H_{-\varepsilon,\varepsilon_3,\ldots,\varepsilon_n}$.

In the first case we have that $(u^\varepsilon x u^{-\varepsilon})(\xi) = u^\varepsilon(x\theta^{-\varepsilon}(y)\xi_0)$. Since $x\theta^{-\varepsilon}(y)\xi_0 \in L^2(M)$, we find that $(u^\varepsilon x u^{-\varepsilon})\xi = \hat{1} \otimes (x\theta^{-\varepsilon}(y)\xi_0) \in H_\varepsilon$. The desired result now follows from the following calculation:

$$(u^\varepsilon x u^{-\varepsilon})\xi = \hat{1} \otimes (x\theta^{-\varepsilon}(y)\xi_0) = \hat{1} \otimes ((\theta^\varepsilon(x)y) \cdot \xi_0)$$
$$= (\hat{1} \cdot (\theta^\varepsilon(x)y)) \otimes \xi_0 = \widehat{\theta^\varepsilon(x)y} \otimes \xi_0$$
$$= \theta^\varepsilon(x)(\hat{y} \otimes \xi_0) = \theta^\varepsilon(x)\xi.$$

The second case follows by exactly the same arguments as the first case.

Finally, in the third case, we also have that $(u^\varepsilon x u^{-\varepsilon})(\xi) = u^\varepsilon(x\theta^{-\varepsilon}(y)\xi_0)$. Since ξ_0 is an element of $(L^2(M) \ominus L^2(N_\varepsilon)) \otimes_N H_{\varepsilon_3,\dots,\varepsilon_n} \subset H_{-\varepsilon,\varepsilon_3,\dots,\varepsilon_n}$ also $x\theta^{-\varepsilon}(y)\xi_0 \in (L^2(M) \ominus L^2(N_\varepsilon)) \otimes_N H_{\varepsilon_3,\dots,\varepsilon_n}$. Therefore $(u^\varepsilon x u^{-\varepsilon})(\xi)$ is equal to $\hat{1} \otimes x\theta^{-\varepsilon}(y)\xi_0 \in H_{\varepsilon,-\varepsilon,\varepsilon_3,\dots,\varepsilon_n}$. The desired result now follows from the following calculation:

$$(u^\varepsilon x u^{-\varepsilon})\xi = \hat{1} \otimes (x\theta^{-\varepsilon}(y)\xi_0) = \hat{1} \otimes ((\theta^\varepsilon(x)y) \cdot \xi_0)$$
$$= (\hat{1} \cdot (\theta^\varepsilon(x)y)) \otimes \xi_0 = \widehat{\theta^\varepsilon(x)y} \otimes \xi_0$$
$$= \theta^\varepsilon(x)(\hat{y} \otimes \xi_0) = \theta^\varepsilon(x)\xi.$$

Altogether, we have showed that $u^\varepsilon x u^{-\varepsilon} = \theta^\varepsilon(x)$ for every $x \in N_\varepsilon$. In particular, $u^\varepsilon u^{-\varepsilon} = 1$. This implies that u^ε is also surjective and hence is a unitary satisfying $(u^\varepsilon)^* = u^{-\varepsilon}$. This justifies the superscript notation. Usually we write u instead of u^1 and call it the *stable unitary*. With this notation we have that

$$uxu^* = \theta(x) \text{ for all } x \in N.$$

This brings us to the definition of $\mathrm{HNN}(M, N, \theta)$.

Definition 1.12.1. The HNN extension $\mathrm{HNN}(M, N, \theta)$ is defined as the von Neumann subalgebra of $B(H)$ generated by M and u:

$$\mathrm{HNN}(M, N, \theta) := \langle M, u\rangle \subset B(H).$$

Let $\Omega = \hat{1} \in L^2(M) \subset H$. Define the normal state on $P = \mathrm{HNN}(M, N, \theta)$ given by $\tau(x) = \langle \Omega, x\Omega\rangle$. It turns out that τ is a faithful normal tracial state for P, see Section 3.3 from [FV12].

Let $P = \mathrm{HNN}(M, N, \theta)$. An element $x \in P$ of the form $x = x_0 u^{\varepsilon_1} x_1 \dots u^{\varepsilon_n} x_n$ with $x_i \in M$ and $\varepsilon_i \in \{-1, 1\}$ is called *reduced* if $x_i \in M \ominus N_{\varepsilon_i}$ whenever $\varepsilon_i \neq \varepsilon_{i+1}$. By convention, when $n = 0$, the reduced elements are the elements $x = x_0$ where $x_0 \in M \ominus \mathbb{C}1$.

Observe that, whenever $x = x_0 u^{\varepsilon_1} x_1 \dots u^{\varepsilon_n} x_n$ is reduced, we have

$$x\Omega = \widehat{x_0} \otimes \dots \otimes \widehat{x_n}.$$

From this we get that $\tau(x) = 0$ whenever x is reduced.

We have the following universal property for HNN extensions.

Theorem 1.12.2 (Proposition 3.2, [FV12])**.** *Let $P := \mathrm{HNN}(M, N, \theta)$ be an* HNN *extension. Assume that (Q, τ_Q) is any tracial von Neumann algebra, that $\pi : M \to Q$ is a trace preserving embedding and that $w \in Q$ is a unitary satisfying*

- $\pi(\theta(x)) = w\pi(x)w^*$ *for all $x \in N$;*
- *for every reduced element $x = x_0 u^{\varepsilon_1} x_1 \dots u^{\varepsilon_n} x_n \in P$, we have that $\tau_Q(\pi(x_0) w^{\varepsilon_1} \dots w^{\varepsilon_n} \pi(x_n)) = 0$.*

Then there exists a unique trace preserving $*$-homomorphism $\tilde{\pi} : P \to Q$ extending π and satisfying $\tilde{\pi}(u) = w$.

The following lemma follows directly from the universal property.

Lemma 1.12.3. *The* HNN *construction behaves well with respect to taking crossed products and tensor products. Concretely,*

- *Let P be a tracial von Neumann algebra. If $\mathrm{HNN}(\Gamma, \Lambda, \theta) \curvearrowright P$ is a trace preserving action, then $P \rtimes \mathrm{HNN}(\Gamma, \Lambda, \theta) = \mathrm{HNN}(P \rtimes \Gamma, P \rtimes \Lambda, \mathrm{Ad}(u_b))$;*
- *If P is a tracial von Neumann algebra, then $P \overline{\otimes} \mathrm{HNN}(M, N, \theta) = \mathrm{HNN}(P \overline{\otimes} M, P \overline{\otimes} N, \mathrm{id} \otimes \theta)$.*

The following proposition states that in the von Neumann algebra case, HNN extensions and amalgamated free products are the same up to amplification. We refer to [Po93] and [VDN92] for all details concerning amalgamated free product von Neumann algebras.

Proposition 1.12.4 ([Ue07, Proposition 3.1])**.** *Let M be a tracial von Neumann algebra, $N \subset M$ a von Neumann subalgebra and $\theta : N \to M$ a trace preserving embedding. Consider the trace preserving embeddings*

$$N \oplus N \hookrightarrow M_2(\mathbb{C}) \overline{\otimes} M : x \oplus y \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & \theta(y) \end{smallmatrix}\right)$$

and

$$N \oplus N \hookrightarrow M_2(\mathbb{C}) \overline{\otimes} N : x \oplus y \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right).$$

Let $u \in \mathrm{HNN}(M, N, \theta)$ be the stable unitary and denote by (e_{ij}), resp. (f_{ij}), the canonical matrix units in $M_2(\mathbb{C})\overline{\otimes}M$, resp. $M_2(\mathbb{C})\overline{\otimes}N$. There is a canonical trace preserving $*$-isomorphism

$$\psi : \mathrm{HNN}(M, N, \theta) \to e_{11}((M_2(\mathbb{C})\overline{\otimes}M) *_{N\oplus N} (M_2(\mathbb{C})\overline{\otimes}N))e_{11}$$

where $\psi(x) = e_{11}x$ for all $x \in M$ and $\psi(u) = e_{12}f_{21}$.

Note that in the amalgamated free product, $e_{11} = f_{11}$ and $e_{22} = f_{22}$. Therefore $e_{12}f_{21}$ is really a unitary.

The following proposition follows from combining [CH08, Theorem 1.1] (as is presented in [Io12, Theorem 6.4]) with Proposition 1.12.4.

Proposition 1.12.5. *Let (M_0, τ) be a tracial von Neumann algebra, $B \subset M_0$ a von Neumann subalgebra and θ a trace preserving embedding of B into M_0. Let $M := \mathrm{HNN}(M_0, B, \theta)$ and $Q \subset pMp$ be a von Neumann subalgebra for some nonzero projection $p \in M$. Then one of the following conditions holds.*

1. *$Q' \cap pMp \prec_M B$;*
2. *$N_{pMp}(Q)'' \prec_M M_0$;*
3. *Qp' is amenable relative to B inside M, for some nonzero projection $p' \in \mathcal{Z}(Q' \cap pMp)$.*

Proof. Put $M_1 = M_2(\mathbb{C}) \otimes M_0$ and $M_2 = M_2(\mathbb{C}) \otimes B$. Consider $B_0 = B \oplus B$ as a subalgebra of both M_1 and M_2, where $B_0 \hookrightarrow M_2$ is diagonal and $B_0 \hookrightarrow M_1$ is given by $b \oplus d \mapsto b \oplus \theta(d)$. Denote by e_{ij} the matrix units in M_1 and by f_{ij} the matrix units in M_2. By Proposition 1.12.4 there is a trace preserving $*$-isomorphism

$$\psi : \mathrm{HNN}(M_0, B, \theta) \to e_{11}(M_1 *_{B_0} M_2)e_{11}$$

where $\psi(x) = e_{11}x$ for all $x \in M_0$ and $\psi(u) = e_{12}f_{21}$. Denote $\mathcal{M} := M_1 *_{B_0} M_2$. Whenever $P \subset M$ is a possibly nonunital inclusion of von Neumann algebras, we have that

(a) $P \prec_M B$ iff $\psi(P) \prec_{\mathcal{M}} B_0$ iff $\psi(P) \prec_{\mathcal{M}} M_2$;

(b) $P \prec_M M_0$ iff $\psi(P) \prec_{\mathcal{M}} M_1$;

(c) P is amenable relative to B inside M iff $\psi(P)$ is amenable relative to B_0 inside $\mathcal{M}$.

Let us check this.

We begin with statement (a). Since $B_0 \subset M_2$ is finite index, we have by Lemma 1.10.3 that $\psi(P) \prec_{\mathcal{M}} B_0$ iff $\psi(P) \prec_{\mathcal{M}} M_2$. On the other hand, $P \prec_M B$ iff $\psi(P) \prec_{\psi(M)} \psi(B)$ iff $\psi(P) \prec_{\mathcal{M}} e_{11}B_0$. So by Lemma 1.10.2, we have that $P \prec_M B$ implies $\psi(B) \prec_{\mathcal{M}} B_0$. For statement (a) to hold, it remains to prove that $\psi(P) \prec_{\mathcal{M}} B_0$ implies $\psi(P) \prec_{\mathcal{M}} e_{11}B_0$. So assume that $\psi(P) \prec_{\mathcal{M}} B_0$. Then

$$\psi(P) \prec_{\mathcal{M}} e_{11}B_0 \text{ or } \psi(P) \prec_{\mathcal{M}} e_{22}B_0.$$

If $\psi(P) \prec_{\mathcal{M}} e_{11}B_0$ holds, then we are done. If $\psi(P) \prec_{\mathcal{M}} e_{22}B_0$ holds, then there exists a nonzero $\psi(P) - e_{22}B_0$-subbimodule $\mathcal{H}$ of $1_{\psi(P)}L^2(\mathcal{M})e_{22}$ with finite $e_{22}B_0$-dimension. Take $v \in M_2(\mathbb{C})$ satisfying $vv^* = e_{22}$ and $v^*v = e_{11}$. Then $\mathcal{H}v$ is a nonzero $\psi(P) - e_{11}B_0$-subbimodule of $1_{\psi(P)}L^2(\mathcal{M})e_{11}$ with finite $e_{11}B_0$-dimension. In other words, $\psi(P) \prec_{\mathcal{M}} e_{11}B_0$. This ends the argument for statement (a).

For statement (b) we have that $P \prec_M M_0$ iff $\psi(P) \prec_{\psi(M)} \psi(M_0)$ iff $\psi(P) \prec_{\mathcal{M}} e_{11}M_1$. Analogous to the previous paragraph, $\psi(P) \prec_{\mathcal{M}} e_{11}M_1$ is equivalent with $\psi(P) \prec_{\mathcal{M}} M_1$. Altogether $P \prec_M M_0$ if and only if $\psi(P) \prec_{\mathcal{M}} M_1$.

Finally for statement (c), we have that P is amenable relative to B inside M if and only if $\psi(P)$ is amenable relative to $\psi(B)$ inside $\psi(M)$. This is then equivalent with $\psi(P)$ being amenable relative to $e_{11}B_0$ inside $e_{11}\mathcal{M}e_{11}$. Since $\langle e_{11}\mathcal{M}e_{11}, e_{e_{11}B_0}\rangle$ is equal to $e_{11}\langle \mathcal{M}, e_{B_0}\rangle e_{11}$, we see using the second characterisation in Definition 1.9.5 that $\psi(P)$ is amenable relative to $e_{11}B_0$ inside $e_{11}\mathcal{M}e_{11}$ if and only if $\psi(P)$ is amenable relative to B_0 inside $\mathcal{M}$. This ends the argument for statement (c).

The proposition now follows immediately from Theorem 6.4 of [Io12]. $\square$

Chapter 2

Partial classification of the Baumslag-Solitar group von Neumann algebras

This chapter will be dedicated to proving the following theorem which partially classifies the Baumslag-Solitar group von Neumann algebras.

Theorem A. *Let* $M = L(\mathrm{BS}(n_1, m_1) \times \ldots \times \mathrm{BS}(n_k, m_k))$ *and* $N = L(\mathrm{BS}(p_1, q_1) \times \ldots \times \mathrm{BS}(p_l, q_l))$*, where* $n_i, m_i, p_j, q_j \in \mathbb{Z}$ *satisfy* $2 \leq n_i < |m_i|$ *and* $2 \leq p_j < |q_j|$ *for every* $i \in \{1, \ldots, k\}$ *and* $j \in \{1, \ldots, l\}$*.*

If there exists a finite index M*-*N*-bimodule* $\mathcal{H}$*, then* $k = l$ *and there exists a permutation* $\sigma \in \mathrm{Sym}(k)$ *such that for every* $i \in \{1, \ldots, k\}$*,*

$$(n_i/|m_i|)^{r_i} = (p_{\sigma(i)}/|q_{\sigma(i)}|)^{s_i}$$

for some $r_i, s_i \in \mathbb{Z}$ *with* $1 \leq r_i, s_i \leq \dim_{M-}(\mathcal{H}) \dim_{-N}(\mathcal{H})$*.*

The following corollary is in fact the main result of [MV13]. It follows immediately from Theorem A using the fact that a stable isomorphism $\alpha : M \to N^t$ yields a finite index M-N-bimodule ${}_M\mathcal{H}(\alpha)_N$ satisfying $\dim_{M-}(\mathcal{H}(\alpha)) \dim_{-N}(\mathcal{H}(\alpha)) = 1$.

Corollary B ([MV13, Theorem A])**.** *Let* $n, m, p, q \in \mathbb{Z}$ *such that* $2 \leq n < |m|$ *and* $2 \leq p < |q|$*. If* $L(\mathrm{BS}(n, m))$ *is stably isomorphic with* $L(\mathrm{BS}(p, q))$*, then* $n/|m| = p/|q|$*.*

Remark. *The proof of Theorem A follows closely the proof of Corollary B as is presented in [MV13]. Apart from some small technicalities, the only real novelty is the notion of finite index correspondence between equivalence relations (see Section 2.2). This notion is needed to deal with finite index bimodules ${}_M\mathcal{H}_N$ instead of stable isomorphisms $\alpha : M \to N^t$.*

We prove our Theorem A by associating a canonical equivalence relation to $L(\mathrm{BS}(n,m))$ and show that it is of type $\mathrm{III}_{n/|m|}$. More precisely, assume that (M,τ) is a von Neumann algebra equipped with a faithful normal tracial state such that $L^2(M,\tau)$ is separable. Whenever $A \subset M$ is an abelian von Neumann subalgebra, the normalizer

$$N_M(A) := \{u \in \mathcal{U}(M) \mid uAu^* = A\}$$

induces a group of trace preserving automorphisms of A. Writing $A = L^\infty(X,\mu)$ with μ being induced by $\tau|_A$, the corresponding orbit equivalence relation is a pmp countable equivalence relation on (X,μ).

More generally, we can consider the set of partial isometries

$$\{u \in M \mid u^*u \text{ and } uu^* \text{ are projections in } A' \cap M \text{ and } uAu^* = Auu^*\} . \tag{2.1}$$

Every such partial isometry induces a partial automorphism of A and hence a partial automorphism of (X,μ). We denote by $\mathcal{R}(A \subset M)$ the equivalence relation generated by all these partial automorphisms. When $A \subset M$ is maximal abelian, i.e. $A = A' \cap M$, then $\mathcal{R}(A \subset M)$ coincides with the orbit equivalence relation induced by the normalizer $N_M(A)$. In particular, in that case the equivalence relation $\mathcal{R}(A \subset M)$ preserves the probability measure μ.

If however $A \subset M$ is not maximal abelian, the partial automorphisms of A induced by the partial isometries in the set (2.1) need not be trace preserving. So in general, $\mathcal{R}(A \subset M)$ can be an equivalence relation of type III.

In Section 2.2 we introduce the notion of finite index correspondence between equivalence relations. Roughly speaking, a finite index correspondence is a stable isomorphism up to finite index between equivalence relations. As was mentioned above, we need this in order to deal with finite index bimodules ${}_M\mathcal{H}_N$ instead of stable isomorphisms $\alpha : M \to N^t$ as is the case in [MV13].

Our key technical result is then Theorem 2.2.6 below, roughly saying the following. Let ${}_M\mathcal{H}_N$ be a nonzero finite index bimodule between II_1 factors M and N. If $A \subset M$ and $B \subset N$ are abelian subalgebras such that $\mathcal{Z}(A' \cap M) = A$ and $\mathcal{Z}(B' \cap N) = B$, and if there exists a nonzero finite index A-B-subbimodule of ${}_A\mathcal{H}_B$, then there must exist a finite index correspondence between the relations $\mathcal{R}(A \subset M)$ and $\mathcal{R}(B \subset N)$ (see Section 2.2 for the exact statement). This implies, as we will see, that if $\mathcal{R}(A \subset M)$ is of type III_λ and $\mathcal{R}(B \subset N)$ is of type III_η, then $\lambda^r = \eta^s$ for some integers r, s with $1 \leq r, s \leq \dim_{M-}(\mathcal{H}) \dim_{-N}(\mathcal{H})$.

In Subsection 2.3.2, we take $M = L(\mathrm{BS}(n,m))$ and A equal to the abelian von Neumann subalgebra generated by the unitary u_a. We prove that $\mathcal{R}(A \subset M)$ is the unique hyperfinite ergodic equivalence relation of type $\mathrm{III}_{n/|m|}$.

The proof of the main theorem now follows the same lines as the proof in [MV13] and can be outlined as follows. We make use of the following notation.

- For all $1 \leq i \leq k$, put $\mathrm{BS}(n_i, m_i) := \langle a_i, b_i \mid b_i a_i^{n_i} b_i^{-1} = a_i^{m_i} \rangle$.
- For all $1 \leq j \leq l$, put $\mathrm{BS}(p_j, q_j) := \langle c_j, d_j \mid d_j c_j^{p_j} d_j^{-1} = c_j^{q_j} \rangle$.

Let M and N be as in the description of Theorem A. Also define

$$M_i = L(\mathrm{BS}(n_1,m_1) \times \ldots \times \mathrm{BS}(n_{i-1},m_{i-1}) \times \langle a_i \rangle \times \mathrm{BS}(n_{i+1},m_{i+1}) \times \ldots \times \mathrm{BS}(n_k,m_k))$$

and

$$N_j = L(\mathrm{BS}(p_1,q_1) \times \ldots \times \mathrm{BS}(p_{j-1},q_{j-1}) \times \langle c_j \rangle \times \mathrm{BS}(p_{j+1},q_{j+1}) \times \ldots \times \mathrm{BS}(p_l,q_l)).$$

Denote by $\mathcal{C}_i$ the centralizer of $\langle a_i^{n_i} \rangle$ inside $\mathrm{BS}(n_i, m_i)$. We first note that $L(\mathcal{C}_i) \subset M$ has no amenable direct summand and that M_i has a finite index subalgebra inside $L(\mathcal{C}_i)' \cap M$. Then we show using [Io12, Theorem 6.4] that if $Q \subset M$ has no amenable direct summand, then there exists an $i \in \{1, \ldots, k\}$ such that $Q' \cap M \prec_M M_i$. This shows that, up to intertwining, the positions of the M_i inside M are canonical. Similarly, the positions of the N_j inside N are canonical. From that one gets that a nonzero finite index M-N-bimodule $\mathcal{H}$ yields both $k = l$ and a permutation $\sigma \in \mathrm{Sym}(k)$ such that $\mathcal{H}$ contains a nonzero finite index M_i-$N_{\sigma(i)}$-subbimodule. It then follows that $\mathcal{H}$ also contains a nonzero finite index $\mathcal{Z}(M_i)$-$\mathcal{Z}(N_{\sigma(i)})$-subbimodule for every i. Using our key technical result, we find that there exists a finite index correspondence between $\mathcal{R}(\mathcal{Z}(M_i) \subset M)$ and $\mathcal{R}(\mathcal{Z}(N_{\sigma(i)}) \subset N)$. On the other hand, these equivalence relations are respectively isomorphic with $\mathcal{R}(L(\langle a \rangle) \subset L(\mathrm{BS}(n_i, m_i)))$ and $\mathcal{R}(L(\langle a \rangle) \subset L(\mathrm{BS}(p_{\sigma(i)}, q_{\sigma(i)})))$. Considering the types of both equivalence relations, one gets the equality $(n_i/|m_i|)^{r_i} = (p_{\sigma(i)}/|q_{\sigma(i)}|)^{s_i}$ for some $r_i, s_i \in \mathbb{Z}$ with $1 \leq r_i, s_i \leq \dim_{M-}(\mathcal{H}) \dim_{-N}(\mathcal{H})$.

2.1 Equivalence relations associated to nonmaximal abelian subalgebras

Let (M, τ) be a tracial von Neumann algebra with $L^2(M, \tau)$ separable. Let $A \subset M$ be an abelian von Neumann subalgebra satisfying $\mathcal{Z}(A' \cap M) = A$. In order to define the equivalence relation $\mathcal{R}(A \subset M)$ that we mentioned before, we need a particular subalgebra of M, that we denote by $\mathrm{Q}_M(A)$. This algebra is closely related to the quasi-normaliser $\mathrm{QN}_M(A)$ of A inside M.

2.1.1 Quasi-regularity and the algebra $\mathrm{Q}_M(A)$

Let (M,τ) be a tracial von Neumann algebra and $N \subset M$ a von Neumann subalgebra. Recall from Section 1.10 that we denote by $\mathrm{QN}_M(N)$ the quasi-normalizer of N inside M, i.e. the unital $*$-algebra defined by

$$\Big\{a \in M \;\Big|\; \exists b_1,\dots,b_k \in M, \exists d_1,\dots,d_r \in M : Na \subset \sum_{i=1}^{k} b_i N,\ aN \subset \sum_{j=1}^{r} N d_j\Big\}.$$

From the same section we also recall that $N \subset M$ is called quasi-regular if $\mathrm{QN}_M(N)'' = M$.

If $A, B \subset M$ are abelian von Neumann subalgebras, we define $\mathrm{Q}_M(A,B)$ as

$$\mathrm{Q}_M(A,B) := \{v \in M \mid vv^* \in A' \cap M\ ,\ v^*v \in B' \cap M \ \text{ and }\ Av = vB\}$$

and we denote $\mathrm{Q}_M(A,A)$ by $\mathrm{Q}_M(A)$.

Whenever $x \in M$ is a normal element (i.e. $xx^* = x^*x$), we denote by $\mathrm{supp}(x)$ its support, i.e. the smallest projection $p \in M$ that satisfies $px = x$ or equivalently, $xp = x$. We have the following well known result that we will implicitly use from this point onward.

Lemma 2.1.1. *Let (M,τ) be a tracial von Neumann algebra and $A \subset M$ an abelian von Neumann subalgebra. Whenever $x \in M$, we have that the projection* $\mathrm{supp}(E_A(xx^*))$ *is the smallest projection $p \in A$ satisfying $px = x$. Similarly, the projection* $\mathrm{supp}(E_A(x^*x))$ *is the smallest projection $p \in A$ satisfying $xp = x$.*

Proof. Fix $x \in M$ and denote by p_x the smallest projection $p \in A$ satisfying $px = x$. Then

$$\begin{aligned} p_x \,\mathrm{supp}(E_A(xx^*)) &= \mathrm{supp}(p_x E_A(xx^*)) \\ &= \mathrm{supp}(E_A(p_x xx^*)) \\ &= \mathrm{supp}(E_A(xx^*)) \end{aligned}$$

and hence, $\mathrm{supp}(E_A(xx^*)) \le p_x$. Let us now show that $p_x \le \mathrm{supp}(E_A(xx^*))$. To that end, note that $(1 - \mathrm{supp}(E_A(xx^*)))E_A(xx^*) = 0$. Applying τ to this equality we get that

$$\begin{aligned}0 &= \tau((1-\mathrm{supp}(E_A(xx^*)))E_A(xx^*)) \\ &= \tau(E_A((1-\mathrm{supp}(E_A(xx^*)))xx^*)) \\ &= \tau((1-\mathrm{supp}(E_A(xx^*)))xx^*) \\ &= ||(1-\mathrm{supp}(E_A(xx^*)))x||_2^2.\end{aligned}$$

Hence $(1-\mathrm{supp}(E_A(xx^*)))x = 0$. This shows that $\mathrm{supp}(E_A(xx^*))x = x$, or equivalently that $p_x \leq \mathrm{supp}(E_A(xx^*))$. Since we already showed that $\mathrm{supp}(E_A(xx^*)) \leq p_x$, we proved that $\mathrm{supp}(E_A(xx^*)) = p_x$.

By replacing x with x^*, we also have that $\mathrm{supp}(E_A(x^*x))$ is the smallest projection $p \in A$ satisfying $xp = x$. □

Whenever $v \in \mathrm{Q}_M(A,B)$, we define $q_v = \mathrm{supp}(E_A(vv^*))$ and $p_v = \mathrm{supp}(E_B(v^*v))$. Let $v \in \mathrm{Q}_M(A,B)$ and $a \in Aq_v$. By definition, there exists an element $b \in B$ such that $av = vb$. Since $v = vp_v$, we may even assume that $b \in Bp_v$. Assume now that $b_1, b_2 \in Bp_v$ satisfy $av = vb_1$ and $av = vb_2$. Then $v^*vb_1 = v^*vb_2$, and therefore $E_B(v^*v)b_1$ equals $E_B(v^*v)b_2$. This then implies that $\mathrm{supp}(E_B(v^*v))b_1 = \mathrm{supp}(E_B(v^*v))b_2$, or equivalently that $p_vb_1 = p_vb_2$. Since $b_1 = p_vb_1$ and $b_2 = p_vb_2$, we have found that $b_1 = b_2$. Altogether, we see that for every $a \in Aq_v$ there exists a unique $b \in Bp_v$ such that $av = vb$. Completely analogous, we have that for every $b \in Bp_v$ there exists a unique $a \in Aq_v$ such that $av = vb$. We denote by $\alpha_v : Aq_v \to Bp_v$ the unique $*$-isomorphism satisfying $av = v\alpha_v(a)$ for all $a \in Aq_v$.

Note that the set $\mathrm{Q}_M(A,B)$ can be $\{0\}$. In Lemma 2.1.3, we will see that $\mathrm{Q}_M(A,B) \neq \{0\}$ if and only if there exists a bifinite A-B-subbimodule ${}_A\mathcal{H}_B$ of ${}_AL^2(M)_B$.

We denote by $\mathrm{PIso}(A,B)$ the set of all partial isomorphisms from A to B, i.e. isomorphisms $\alpha : Aq \to Bp$, where $q \in A$ and $p \in B$ are projections. We write $\mathrm{PAut}(A)$ instead of $\mathrm{PIso}(A,A)$. Note that to every $\alpha \in \mathrm{PIso}(A,B)$ we can associate an A-B-bimodule ${}_A\mathcal{H}(\alpha)_B$ given by $\mathcal{H}(\alpha) = L^2(Bp)$ and $a\xi b = \alpha(aq)\,\xi\,bp$. The composition of two partial isomorphisms is defined as follows: if $\alpha \in \mathrm{PIso}(B,C)$ and $\beta \in \mathrm{PIso}(A,B)$ are given by $\alpha : Bp \to Cr$ and $\beta : Aq \to Bp'$ for projections $q \in A$, $p, p' \in B$ and $r \in C$, then the composition $\alpha \circ \beta \in \mathrm{PIso}(A,C)$ is defined by $x \mapsto \alpha(\beta(x))$ for all $x \in Aq\beta^{-1}(pp')$.

Lemma 2.1.2. *Let A and B be abelian von Neumann algebras. Then every bifinite A-B-bimodule ${}_A\mathcal{H}_B$ is isomorphic to a direct sum of bimodules of the form ${}_A\mathcal{H}(\alpha)_B$ with $\alpha \in \mathrm{PIso}(A,B)$.*

Proof. By a standard maximality argument, it suffices to show that ${}_A\mathcal{H}_B$ contains, up to isomorphism, a nonzero A-B-subbimodule of the form ${}_A\mathcal{H}(\alpha)_B$.

By Proposition F.10 of [BO08], we have that there exists a nonzero projection $q \in A$, a nonzero projection $p \in B$ and a nonzero Aq-B-subbimodule $\mathcal{K}$ of $q\mathcal{H}$ such that $\mathcal{K}$ is isomorphic, as a right B-module, with $L^2(Bp)$. Since $\mathcal{K}$ is also a left Aq-module with finite Aq-dimension, there exists a normal $*$-homomorphism $\psi : Aq \to Bp$ such that $\psi(Aq) \subset Bp$ is finite index and

$$ {}_{Aq}\mathcal{K}_B \cong {}_{Aq}\mathcal{H}(\psi)_B, $$

where $\mathcal{H}(\psi) = L^2(Bp)$ and $a\xi b = \psi(aq)\,\xi\,bp$. Since $\psi(Aq) \subset Bp$ is a finite index inclusion of abelian von Neumann algebras, there exists a nonzero projection $z \in Bp$ such that $\psi(Aq)z = Bz$. Write $\tilde{\psi} : Aq \to Bz : x \mapsto \psi(x)z$. Then $\tilde{\psi}$ is surjective. Put $I := \ker(\tilde{\psi})$. Then I is a two-sided ideal of Aq that is weakly closed. By [Tak79, Proposition II.3.12], there exists a projection $r \in Aq$ such that $I = Ar$. Simply putting $\alpha : A(q-r) \to Bz : x \mapsto \tilde{\psi}(x)$, we see that $\alpha \in \mathrm{PIso}(A,B)$ and ${}_A\mathcal{H}(\alpha)_B$ is isomorphic with a subbimodule of ${}_A\mathcal{H}_B$. This ends the proof. □

With this lemma, we are able to prove the following.

Lemma 2.1.3. *Let (M,τ) be a tracial von Neumann algebra and let $A, B \subset M$ abelian von Neumann subalgebras. Then the following statements hold.*

1. *If $\alpha \in \mathrm{PIso}(A,B)$ and if $\theta : {}_A\mathcal{H}(\alpha)_B \to {}_AL^2(M)_B$ is an A-B-bimodular isometry, then there exists a partial isometry $v \in \mathrm{Q}_M(A,B)$ such that $\alpha = \alpha_v$ and such that*
$$ \theta(\mathcal{H}(\alpha)) \subset \overline{v(B' \cap M)}^{\|\cdot\|_2} \subset \overline{\mathrm{span}}^{\|\cdot\|_2}\, \mathrm{Q}_M(A,B)\ . $$
2. *Every bifinite A-B-subbimodule ${}_A\mathcal{H}_B$ of ${}_AL^2(M)_B$ is contained in $\overline{\mathrm{span}}^{\|\cdot\|_2}\, \mathrm{Q}_M(A,B)$.*
3. *$\mathrm{Q}_M(A)'' = \mathrm{QN}_M(A)''$.*
4. *We have $\mathrm{Q}_M(A,B) \neq \{0\}$ if and only if ${}_AL^2(M)_B$ admits a nonzero bifinite A-B-subbimodule.*

Proof. 1. Let $\alpha : Aq \to Bp$ be an element of $\mathrm{PIso}(A,B)$. Define $\xi := \theta(p) \in L^2(M)$ and let $\xi = v|\xi|$ be its polar decomposition, i.e. the unique partial isometry $v \in M$ and the unique element $|\xi|$ of the $\|\cdot\|_2$-closure of M^+ such that $\xi = v|\xi|$ and v^*v is the smallest projection $p \in M$ such that $p|\xi| = |\xi|$. For all $a \in A$, we have $a\xi = \xi\alpha(a)$ and hence, $av = v\alpha(a)$. Furthermore

$p = \operatorname{supp}(E_B(v^*v))$ and $q = \alpha^{-1}(p) = \operatorname{supp}(E_A(vv^*))$. So we find that $v \in \mathrm{Q}_M(A,B)$ and $\alpha = \alpha_v$. Because $|\xi| \in L^2(B' \cap M)$, we have that $\xi = v|\xi|$ is an element of the $||\cdot||_2$-closure of $v(B' \cap M)$. Since p generates ${}_A\mathcal{H}(\alpha)_B$ as a right Hilbert B-module, we have proven the first inclusion in statement 1. Since $v \in \mathrm{Q}_M(A,B)$, also $v(B' \cap M) \subset \mathrm{Q}_M(A,B)$ and the second inclusion in statement 1 is proven as well.

2. Let ${}_A\mathcal{H}_B$ be a bifinite A-B-subbimodule of ${}_AL^2(M)_B$. By Lemma 2.1.2, ${}_A\mathcal{H}_B$ is isomorphic to a direct sum of bimodules of the form ${}_A\mathcal{H}(\alpha_i)_B$ with $\alpha_i \in \mathrm{PIso}(A,B)$. Using statement 1 of the lemma, we find that $\mathcal{H}$ is generated by subspaces of $\overline{\operatorname{span}}^{||\cdot||_2}\, \mathrm{Q}_M(A,B)$. This proves statement 2.

3. By definition, we have $\mathrm{Q}_M(A)'' \subset \mathrm{QN}_M(A)''$. On the other hand, by considering the bifinite A-A-bimodules ${}_A(\overline{\operatorname{span}}^{||\cdot||_2} AvA)_A$ for $v \in \mathrm{QN}_M(A)$, we have that ${}_AL^2(\mathrm{QN}_M(A)'')_A$ is a direct sum of bifinite A-A-subbimodules of ${}_AL^2(M)_A$. So by statement 2, we have that $L^2(\mathrm{QN}_M(A)'') \subset \overline{\operatorname{span}}^{||\cdot||_2}(\mathrm{Q}_M(A))$. Since the $||\cdot||_2$-norm topology and the strong operator topology are the same on bounded sets, we find using the Kaplansky density theorem (see e.g. [Tak79, Theorem II.4.8]), that $\mathrm{QN}_M(A)'' = \mathrm{Q}_M(A)''$.

Finally, 4 is an immediate consequence of 2. ☐

We end this subsection with the following two lemmas, clarifying why later, we will consider abelian subalgebras $A \subset M$ satisfying $\mathcal{Z}(A' \cap M) = A$. Note that since A is abelian, the condition $\mathcal{Z}(A' \cap M) = A$ is equivalent with the 'bicommutant' property $(A' \cap M)' \cap M = A$.

Lemma 2.1.4. *Let (M,τ) be a tracial von Neumann algebra and $A \subset M$ an abelian von Neumann subalgebra satisfying $\mathcal{Z}(A' \cap M) = A$. Whenever $x \in M$, the projection $\operatorname{supp}(E_A(xx^*))$ is equal to the projection of $L^2(M)$ onto the closed linear span of $(A' \cap M)xM \subset L^2(M)$.*

Proof. Fix $x \in M$ and denote by p_x the projection of $L^2(M)$ onto the closed linear span of $(A' \cap M)xM \subset L^2(M)$. Then $p_x \in \mathrm{B}(L^2(M))$ commutes with the left $(A' \cap M)$-action and with the right M-action. Hence $p_x \in (A' \cap M)' \cap M = A$. Also $p_x x = x$, since $xL^2(M)$ lies inside the closed linear span of $(A' \cap M)xM \subset L^2(M)$. Altogether p_x is a projection in A satisfying $p_x x = x$. Equivalently, $\operatorname{supp}(E_A(xx^*)) \leq p_x$ by Lemma 2.1.1.

On the other hand, whenever $a \in A' \cap M$ and $y \in M$, we have that

$$\operatorname{supp}(E_A(xx^*))axy = a\operatorname{supp}(E_A(xx^*))xy = axy.$$

So $\operatorname{supp}(E_A(xx^*))p_x = p_x$, or equivalently $p_x \leq \operatorname{supp}(E_A(xx^*))$. Since we already showed that $\operatorname{supp}(E_A(xx^*)) \leq p_x$, we are done. ☐

Lemma 2.1.5. *Let (M,τ) be a tracial von Neumann algebra and $A, B, C \subset M$ abelian von Neumann subalgebras. If $v \in \mathrm{Q}_M(A,B)$, $w \in \mathrm{Q}_M(B,C)$ and if $\mathcal{Z}(B' \cap M) = B$, then there exists an element $u \in \mathrm{Q}_M(A,C)$ such that $\alpha_w \circ \alpha_v = \alpha_u$.*

Proof. Choose $v \in \mathrm{Q}_M(A,B)$ and $w \in \mathrm{Q}_M(B,C)$. We first show that $vbw \in \mathrm{Q}_M(A,C)$ for every $b \in B' \cap M$.

Let $b \in B' \cap M$ and define $q_{vbw} = \mathrm{supp}(E_A((vbw)(vbw)^*))$. Since $\alpha_v^{-1}(p_v q_w) \in A$ satisfies $\alpha_v^{-1}(p_v q_w) vbw = v p_v q_w bw = v q_w bw = vb q_w w = vbw$, we have that $q_{vbw} \leq \alpha_v^{-1}(p_v q_w)$. Or equivalently, $A q_{vbw} \subset \mathrm{dom}(\alpha_w \circ \alpha_v)$. On the other hand, for every $a \in A q_{vbw}$ we have that

$$avbw = v\alpha_v(a)bw = vb\alpha_v(a)w = vbw\alpha_w(\alpha_v(a)).$$

This shows that vbw is indeed an element of $\mathrm{Q}_M(A,C)$ and that $\alpha_{vbw} = \alpha_w \circ \alpha_v|_{Aq_{vbw}}$. We claim that $\bigvee_{b \in B' \cap M} q_{vbw} = \alpha_v^{-1}(q_w p_v)$.

Denote $\alpha_v^{-1}(q_w p_v) - \bigvee_{b \in B' \cap M} q_{vbw}$ by r. We need to prove that r is zero. Since $(B' \cap M)' \cap M = B$, we have by Lemma 2.1.4 that for every $x \in M$, the projection $\mathrm{supp}(E_B(xx^*))$ equals the projection of $L^2(M)$ onto the closed linear span of $(B' \cap M)xM \subset L^2(M)$. Since $w^* b v^* r = 0$ for every $b \in B' \cap M$, it follows that $w^* q_{v^* r} = 0$. Therefore q_w is orthogonal to $q_{v^* r}$. Because $q_{v^* r} = \alpha_v(r)$ and $\alpha_v(r) \leq q_w$, it follows that $\alpha_v(r) = 0$. Hence $r = 0$ and our claim that $\bigvee_{b \in B' \cap M} q_{vbw} = \alpha_v^{-1}(q_w p_v)$ is proven.

Observe that if b is an element of $B' \cap M$ and $q \in A q_{vbw}$ is a projection, then $\alpha_v(q)b$ is an element of $B' \cap M$ and $q = q q_{vbw} = q_{qvbw} = q_{v\alpha_v(q)bw}$. So, by cutting down with appropriate projections, we find $b_n \in B' \cap M$ such that the projections $q_{vb_n w}$ are orthogonal and sum up to $\alpha_v^{-1}(q_w p_v)$. In particular, the left supports, resp. right supports, of the elements $vb_n w$ are orthogonal. Finally replacing b_n by $b_n / ||vb_n w||$, we can define $u = \sum_n vb_n w$. It follows that $u \in \mathrm{Q}_M(A,C)$ and $\alpha_w \circ \alpha_v = \alpha_u$. □

2.1.2 Equivalence relations associated to nonmaximal abelian subalgebras

Throughout this subsection, we fix a tracial von Neumann algebra (M,τ) with $L^2(M,\tau)$ separable. We also fix an abelian von Neumann subalgebra $A \subset M$ satisfying $\mathcal{Z}(A' \cap M) = A$. Since A is abelian, we can choose a standard probability space (X,μ) such that $A = L^\infty(X,\mu)$. For every nonsingular partial automorphism φ of (X,μ), we denote by α_φ the corresponding partial automorphism of A. Similarly, for every partial automorphism α of A, we denote by φ_α the corresponding nonsingular automorphism of (X, μ).

The next proposition shows that $\mathrm{Q}_M(A)$ induces a nonsingular countable equivalence relation $\mathcal{R}(A \subset M)$ on (X, μ). For it, we need to introduce the notation

$$\mathcal{G}(A \subset M) := \{\alpha_v \mid v \in \mathrm{Q}_M(A)\} . \tag{2.2}$$

Proposition 2.1.6. *There exists a nonsingular countable equivalence relation $\mathcal{R}$ on (X, μ) with the following property: a nonsingular partial automorphism φ of X satisfies $\alpha_\varphi \in \mathcal{G}(A \subset M)$ if and only if $(x, \varphi(x)) \in \mathcal{R}$ for a.e. $x \in \mathrm{dom}(\varphi)$.*

Moreover, $\mathcal{R}$ is essentially unique: if a nonsingular countable equivalence relation $\mathcal{R}'$ on (X, μ) satisfies the same property, then there exists a subset $X_0 \subset X$ with $\mu(X \setminus X_0) = 0$ and $\mathcal{R}|_{X_0} = \mathcal{R}'|_{X_0}$.

We denote $\mathcal{R}(A \subset M) := \mathcal{R}$. The equivalence relation $\mathcal{R}(A \subset M)$ is ergodic if and only if $\mathrm{QN}_M(A)''$ is a factor.

Before proving Proposition 2.1.6, we introduce some terminology and a lemma. To every $\alpha \in \mathrm{PAut}(A)$ are associated the support projections $q_\alpha, p_\alpha \in A$ such that $\alpha : Aq_\alpha \to Ap_\alpha$ is a $*$-isomorphism. Assume that $\alpha \in \mathrm{PAut}(A)$ and $\mathcal{F} \subset \mathrm{PAut}(A)$. We say that α is a *gluing* of elements in $\mathcal{F}$, if there exists a sequence of elements $\alpha_n \in \mathcal{F}$ and projections $q_n \in A$ such that $q_\alpha = \sum_n q_n$ and such that $q_n \leq q_{\alpha_n}$ and $\alpha|_{Aq_n} = \alpha_n|_{Aq_n}$ for all n.

Lemma 2.1.7. *Let $\mathcal{J} \subset \mathrm{Q}_M(A)$ and $v \in \mathrm{Q}_M(A)$ such that $v \in \overline{\mathrm{span}}^{\|\cdot\|_2}\mathcal{J}$. Then α_v is a gluing of elements in $\{\alpha_w \mid w \in \mathcal{J}\}$.*

Proof. By a standard maximality argument, it suffices to prove that for every nonzero projection $q \in Aq_v$, there exists a nonzero subprojection $q_0 \in Aq$ and a $w \in \mathcal{J}$ such that $q_0 \leq q_w$ and $\alpha_v|_{Aq_0} = \alpha_w|_{Aq_0}$.

So fix a nonzero projection $q \in Aq_v$. It follows that $qE_A(vv^*) \neq 0$. Since $v \in \overline{\mathrm{span}}^{\|\cdot\|_2}\mathcal{J}$, we can pick a $w \in \mathcal{J}$ such that $qE_A(vw^*) \neq 0$. Define $q_1 := \mathrm{supp}(E_A(vw^*))$ and note that $q_1 \in q_v A q_w = Aq_v q_w$. Also note that $qq_1 \neq 0$. For all $a \in A$, we have

$$\alpha_v^{-1}(ap_v)\, vw^* = v\, a\, w^* = vw^*\, \alpha_w^{-1}(ap_w) \; .$$

Applying the conditional expectation onto A and using that A is abelian, we find that

$$\alpha_v^{-1}(ap_v)\, q_1 = \alpha_w^{-1}(ap_w)\, q_1 \quad \text{for all } \; a \in A \; .$$

This means that $\alpha_v|_{Aq_1} = \alpha_w|_{Aq_1}$. We put $q_0 := qq_1$. We already showed that $q_0 \neq 0$. Since $q_0 \leq q_1$, we have that $\alpha_v|_{Aq_0} = \alpha_w|_{Aq_0}$. $\square$

Proof of Proposition 2.1.6. We say that a subpseudogroup $\mathcal{G} \subset \mathrm{PAut}(A)$ is *of countable type* if there exists a countable subset $\mathcal{J} \subset \mathcal{G}$ such that every $\alpha \in \mathcal{G}$ is a gluing of elements in $\mathcal{J}$. To prove the first part of the proposition, we first show that $\mathcal{G}(A \subset M)$ is a subpseudogroup of countable type of $\mathrm{PAut}(A)$. From Lemma 2.1.5, it follows that $\mathcal{G}(A \subset M)$ is indeed a subpseudogroup. Since M is separable for the $||.||_2$-norm topology, we can choose a countable $\|\cdot\|_2$-dense subset $\mathcal{J} \subset \mathrm{Q}_M(A)$. By Lemma 2.1.7, every $\alpha \in \mathcal{G}(A \subset M)$ is a gluing of elements in $\{\alpha_w \mid w \in \mathcal{J}\}$. Hence $\mathcal{G}(A \subset M)$ is of countable type. Now define $\mathcal{R}_0 := \bigcup_{v \in \mathcal{J}} \mathrm{graph}(\varphi_{\alpha_v})$. Since $\mathcal{J}$ is a countable set, there exists a conegligible subset $U \subset X$ such that $\mathcal{R} := \mathcal{R}_0 \cap (U \times U)$ is an equivalence relation. By construction $\mathcal{R}$ is a nonsingular countable equivalence relation on (X, μ) with the property that a nonsingular partial automorphism φ of X satisfies $\alpha_\varphi \in \mathcal{G}(A \subset M)$ if and only if $(x, \varphi(x)) \in \mathcal{R}$ for a.e. $x \in \mathrm{dom}(\varphi)$.

Let $\mathcal{R}'$ be another nonsingular countable equivalence relation on (X, μ) satisfying the same property. For $n \in \mathbb{N}$, we fix $\varphi_n \in [\mathcal{R}]$ such that $\mathcal{R} = \bigcup_{n \in N} \mathrm{graph}(\varphi_n)$. Since $\mathrm{graph}\,\varphi_n \subset \mathcal{R}$, we have that $\alpha_{\varphi_n} \in \mathcal{G}(A \subset M)$. But then $(x, \varphi_n(x)) \in \mathcal{R}'$ for a.e. $x \in X$. So, up to a conegligible restriction, we have that $\mathcal{R} \subset \mathcal{R}'$. By symmetry, we also have that $\mathcal{R}' \subset \mathcal{R}$ up to a conegligible restriction. Hence we can conclude that $\mathcal{R}$ and $\mathcal{R}'$ essentially coincide.

It remains to prove that $\mathcal{R}$ is ergodic if and only if $\mathrm{QN}_M(A)''$ is a factor. Since $A' \cap M \subset \mathrm{QN}_M(A)''$ and since we assumed that $(A' \cap M)' \cap M = A$, the center of $\mathrm{QN}_M(A)''$ is a subalgebra of A. By Lemma 2.1.3.(3), we have $\mathrm{QN}_M(A)'' = \mathrm{Q}_M(A)''$. Therefore,

$$\mathcal{Z}(\mathrm{QN}_M(A)'') = \{a \in A \mid av = va \text{ for all } v \in \mathrm{Q}_M(A)\} .$$

The right hand side equals $A^{\mathcal{R}}$, the subalgebra of $\mathcal{R}$-invariant functions in A. So $\mathcal{R}$ is ergodic if and only if $\mathrm{QN}_M(A)''$ is a factor. $\square$

The following lemma will allow us to easily compute $\mathcal{R}(A \subset M)$ in concrete examples.

Lemma 2.1.8. *Let (M, τ) be a tracial von Neumann algebra and $A \subset M$ an abelian von Neumann subalgebra satisfying $\mathcal{Z}(A' \cap M) = A$. Let $\mathcal{F} \subset M$ be a subset such that*

- $M = (\mathcal{F} \cup \mathcal{F}^* \cup (A' \cap M))''$,
- *as an A-A-bimodule, $\overline{\mathrm{span}}^{||\cdot||_2} A\mathcal{F}A$ is isomorphic to a direct sum of bimodules of the form ${}_A\mathcal{H}(\alpha_n)_A$ with $\alpha_n \in \mathrm{PAut}(A)$.*

Choose nonsingular partial automorphisms φ_n of (X,μ) such that $\alpha_n = \alpha_{\varphi_n}$ for all n. Up to conegligible restriction, $\mathcal{R}(A \subset M)$ is generated by the graphs of the partial automorphisms φ_n.

Proof. We again use the notation (2.2). By Lemma 2.1.3.(1), we find $v_n \in \mathrm{Q}_M(A)$ such that $\alpha_n = \alpha_{v_n}$ and

$$\overline{\operatorname{span}}^{\|\cdot\|_2} A\mathcal{F}A \subset \overline{\operatorname{span}}^{\|\cdot\|_2}\{v_n(A' \cap M) \mid n \in \mathbb{N}\} . \tag{2.3}$$

In particular, we have $\alpha_n \in \mathcal{G}(A \subset M)$. Choose nonsingular partial automorphisms φ_n of (X,μ) such that $\alpha_n = \alpha_{\varphi_n}$ for all n.

Denote by $\mathcal{R}$ the smallest (up to conegligible restriction) equivalence relation on (X,μ) that contains the graphs of all the partial automorphisms φ_n. By the previous paragraph, we know that $\mathcal{R}$ is a subequivalence relation of $\mathcal{R}(A \subset M)$. Denote by $\mathcal{J}$ the set of all products of elements in

$$\{v_n \mid n \in \mathbb{N}\} \cup \{v_n^* \mid n \in \mathbb{N}\} \cup (A' \cap M) .$$

By construction, the graph of every α_w, $w \in \mathcal{J}$, belongs to $\mathcal{R}$. Combining our assumption that $M = (\mathcal{F} \cup \mathcal{F}^* \cup (A' \cap M))''$ with (2.3), it follows that $\operatorname{span}\mathcal{J}$ is $\|\cdot\|_2$-dense in $L^2(M)$. By Lemma 2.1.7, every $\alpha \in \mathcal{G}(A \subset M)$ is a gluing of elements in $\{\alpha_w \mid w \in \mathcal{J}\}$. So the graph of every $\alpha \in \mathcal{G}(A \subset M)$ belongs to $\mathcal{R}$ a.e. Hence $\mathcal{R}$ equals $\mathcal{R}(A \subset M)$ almost everywhere. □

We finally note in the following proposition that every nonsingular countable equivalence relation $\mathcal{R}$ arises as $\mathcal{R}(A \subset M)$.

Proposition 2.1.9. *Let $\mathcal{R}$ be a nonsingular countable equivalence relation. Then there exists a quasi-regular inclusion of an abelian von Neumann algebra A in a tracial von Neumann algebra (M,τ) satisfying $\mathcal{Z}(A' \cap M) = A$ and such that $\mathcal{R} \cong \mathcal{R}(A \subset M)$.*

Proof. Let $\mathcal{R}$ be a nonsingular countable equivalence relation on a standard probability space (X,μ). Denote by (P,Tr) the unique hyperfinite II_∞ factor, i.e. the unique amenable semifinite factor (P,Tr) such that $\mathrm{Tr}(1) = \infty$ and P contains no minimal projections. Choose a trace-scaling action $(\alpha_t)_{t\in\mathbb{R}}$ of $\mathbb{R}$ on P, meaning that $\mathrm{Tr}\circ\alpha_t = e^{-t}\,\mathrm{Tr}$ (see e.g. [Dy95, Theorem 4.2]). The corresponding action of $\mathbb{R}$ on $L^2(P)$ will also be denoted by (α_t). We denote by $\omega : \mathcal{R} \to \mathbb{R}$ the Radon-Nikodym 1-cocycle of $\mathcal{R}$ (see Section 1.5).

In the same way as with the Maharam extension of a nonsingular group action, the group $[\mathcal{R}]$ admits a natural trace preserving action on $L^\infty(X) \mathbin{\overline{\otimes}} P = L^\infty(X,P)$. Concretely, this action is given by

$$(\varphi \cdot F)(x) := \alpha_{\omega(x,\varphi^{-1}(x))}(F(\varphi^{-1}(x))),$$

where $\varphi \in [\mathcal{R}]$ and $F \in L^\infty(X, P)$. We denote by $(\mathcal{M}, \mathrm{Tr})$ the crossed product of this action. For completeness, we recall the construction of $(\mathcal{M}, \mathrm{Tr})$.

To every $\varphi \in [[\mathcal{R}]]$, we associate the operator W_φ on $L^2(\mathcal{R}, L^2(P))$ given by

$$(W_\varphi \xi)(x, y) = \begin{cases} \alpha_{\omega(x, \varphi^{-1}(x))}(\xi(\varphi^{-1}(x), y)) & \text{if } x \in \mathrm{dom}(\varphi^{-1}) \, , \\ 0 & \text{otherwise,} \end{cases}$$

for every $\xi \in L^2(\mathcal{R}, L^2(P))$. One checks that $W_\varphi W_\psi = W_{\varphi \circ \psi}$ and $W_\varphi^* = W_{\varphi^{-1}}$.

Furthermore, we represent $L^\infty(X) \mathbin{\overline{\otimes}} P = L^\infty(X, P)$ on $L^2(\mathcal{R}, L^2(P))$ by

$$(F\xi)(x, y) = F(x)\xi(x, y) \text{ for all } \xi \in L^2(\mathcal{R}, L^2(P)) \text{ and } F \in L^\infty(X, P) \, .$$

Note that the partial isometries W_φ, $\varphi \in [[\mathcal{R}]]$, normalize $L^\infty(X, P)$ and that

$$(W_\varphi^* F W_\varphi)(x) = \begin{cases} \alpha_{\omega(x, \varphi(x))}(F(\varphi(x))) & \text{if } x \in \mathrm{dom}\, \varphi \\ 0 & \text{otherwise.} \end{cases}$$

Define $\mathcal{M}$ as the von Neumann algebra generated by $L^\infty(X, P)$ and the partial isometries W_φ, $\varphi \in [[\mathcal{R}]]$. Denote by $\Delta \subset \mathcal{R}$ the diagonal subset. Fix $T \in \mathcal{M}$ and define on $L^2(\Delta, L^2(P))$ the bounded sesquilinear form

$$(\xi, \eta)_T := \langle T\xi, \eta \rangle.$$

Then there exists a unique element $E(T) \in \mathrm{B}(L^2(\Delta, L^2(P)))$ such that

$$\langle T\xi, \eta \rangle = \langle E(T)\xi, \eta \rangle,$$

for every $\xi, \eta \in L^2(\Delta, L^2(P))$. We show that E is a normal faithful conditional expectation from $\mathcal{M}$ onto $L^\infty(X, P)$.

In order to do that, we associate to every $\varphi \in [[\mathcal{R}]]$ the operator V_φ on $L^2(\mathcal{R}, L^2(P))$ given by

$$(V_\varphi \xi)(x, y) = \begin{cases} \xi(x, \varphi^{-1}(y)) & \text{if } y \in \mathrm{dom}(\varphi^{-1}) \, , \\ 0 & \text{otherwise,} \end{cases}$$

for every $\xi \in L^2(\mathcal{R}, L^2(P))$. Furthermore, we denote by ρ the representation of $L^\infty(X, P)$ on $L^2(\mathcal{R}, L^2(P))$ given by

$$(\rho(F)\xi)(x, y) = \xi(x, y)\alpha_{\omega(x,y)}(F(y)) \text{ for all } \xi \in L^2(\mathcal{R}, L^2(P)), F \in L^\infty(X, P).$$

Note that $\rho(F)$ and V_φ commute with $\mathcal{M}$ for every $F \in L^\infty(X, P)$ and $\varphi \in [[\mathcal{R}]]$.

We have for every $T \in \mathcal{M}$, $F \in L^\infty(X, P)$ and $\xi, \eta \in L^2(\Delta, L^2(P))$ that

$$\begin{aligned}\langle E(T)\rho(F)\xi, \eta\rangle &= \langle T\rho(F)\xi, \eta\rangle = \langle \rho(F)T\xi, \eta\rangle = \langle T\xi, \rho(F)^*\eta\rangle \\ &= \langle E(T)\xi, \rho(F)^*\eta\rangle = \langle \rho(F)E(T)\xi, \eta\rangle\end{aligned}$$

Therefore $E(T)$ commutes with $\rho(L^\infty(X, P))$, and hence is an element of $L^\infty(X, P)$. So E is a linear map from $\mathcal{M}$ to $L^\infty(X, P)$.

We also have that

- E is positive;
- $E(F) = F$ for every $F \in L^\infty(X, P)$;
- E is $L^\infty(X, P)$-$L^\infty(X, P)$-bimodular.

To show that E is faithful, let $T \in \mathcal{M}$ satisfy $E(T^*T) = 0$. Then for every $\xi \in L^2(\Delta, L^2(P))$:

$$0 = \langle E(T^*T)\xi, \xi\rangle = \langle T^*T\xi, \xi\rangle = ||T\xi||^2.$$

Hence $T\xi = 0$ for every $\xi \in L^2(\Delta, L^2(P))$. But then $TV_\varphi\xi = 0$ for every $\varphi \in [[\mathcal{R}]]$ and every $\xi \in L^2(\Delta, L^2(P))$. Since $V_\varphi(L^2(\Delta, L^2(P)))$ is equal to $L^2(\text{graph}(\varphi), L^2(P))$, we see that T must be zero on the whole of $L^2(\mathcal{R}, L^2(P))$. This shows that E is indeed faithful.

To check that E is normal, let T_i be an increasing bounded net of positive elements of $\mathcal{M}$. Then $E(\sup_i T_i) \geq \sup_i E(T_i)$ and we have that

$$\begin{aligned}\langle E(\sup_i T_i)\xi, \xi\rangle &= \langle (\sup_i T_i)\xi, \xi\rangle = \sup_i \langle T_i\xi, \xi\rangle \\ &= \sup_i \langle E(T_i)\xi, \xi\rangle = \langle (\sup_i E(T_i))\xi, \xi\rangle.\end{aligned}$$

Hence $E(\sup_i T_i) = \sup_i E(T_i)$, showing that E is normal.

Altogether, we conclude that E is indeed a normal faithful conditional expectation from $\mathcal{M}$ onto $L^\infty(X, P)$. The formula $\text{Tr} := (\mu \otimes \text{Tr}) \circ E$ now defines a normal faithful semi-finite trace on $\mathcal{M}$.

Let us now show that the relative commutant $L^\infty(X)' \cap \mathcal{M}$ equals $L^\infty(X) \mathbin{\overline{\otimes}} P$. To that end, let $T \in L^\infty(X)' \cap \mathcal{M}$. Fix $\xi \in L^2(\Delta, L^2(P))$ and set $f = T\xi \in L^2(\mathcal{R}, L^2(P))$. Then, for every $F \in L^\infty(X)$,

$$Ff = FT\xi = TF\xi = T\rho(F)\xi = \rho(F)T\xi = \rho(F)f.$$

Therefore $f \in L^2(\mathcal{R}, L^2(P))$ is supported on the diagonal Δ. Write $S := T|_{L^2(\Delta, L^2(P))}$, then we just showed that S is an element of $\mathrm{B}(L^2(\Delta, L^2(P)))$. Since T commutes with $\rho(L^\infty(X,P))$, we moreover have that $S \in L^\infty(X) \otimes P$. Now T also commutes with V_φ for every $\varphi \in [[\mathcal{R}]]$, and hence

$$SV_\varphi \xi = V_\varphi S\xi = V_\varphi T\xi = TV_\varphi \xi,$$

for every $\varphi \in [[\mathcal{R}]]$ and $\xi \in L^2(\Delta, L^2(P))$. Since $V_\varphi(L^2(\Delta, L^2(P)))$ is equal to $L^2(\mathrm{graph}(\varphi), L^2(P))$, we see that T and S coincide on the whole of $L^2(\mathcal{R}, L^2(P))$. Therefore $T \in L^\infty(X) \otimes P$ and we conclude that $L^\infty(X)' \cap \mathcal{M}$ is indeed equal to $L^\infty(X) \mathbin{\overline{\otimes}} P$.

Fix a nonzero projection $q \in P$ with $\mathrm{Tr}(q) = 1$. Define the projection $p \in L^\infty(X) \mathbin{\overline{\otimes}} P$ given by $p = 1 \otimes q$ and write $M := p\mathcal{M}p$. Then the restriction of Tr to M gives a normal faithful tracial state τ on M. Let $A := L^\infty(X)p$. For every $F \in L^\infty(X,P)$ and $\varphi \in [[\mathcal{R}]]$, we have that

$$\begin{aligned}(pFW_\varphi p)A &= pFW_\varphi L^\infty(X)p = pFL^\infty(X)W_\varphi p \\ &= pL^\infty(X)FW_\varphi p = A(pFW_\varphi p).\end{aligned}$$

Hence A is a quasi-regular abelian von Neumann subalgebra of M. Since P is a factor and $L^\infty(X)' \cap \mathcal{M} = L^\infty(X) \mathbin{\overline{\otimes}} P$, it also follows that

$$\mathcal{Z}(A' \cap M) = \mathcal{Z}(L^\infty(X) \mathbin{\overline{\otimes}} qPq) = L^\infty(X) \mathbin{\overline{\otimes}} \mathbb{C}q = A.$$

We finally prove that $\mathcal{R} \cong \mathcal{R}(A \subset M)$. Write $\mathcal{R} = \bigcup_k \mathrm{graph}(\varphi_k)$, with $\varphi_k \in [\mathcal{R}]$. Then φ_k induces an automorphism of $L^\infty(X)$ and hence of $A = L^\infty(X)p$ that we denote by $\beta_k \in \mathrm{Aut}(A)$. Since P is a II_∞ factor and $q \in P$ is a finite projection, we can choose partial isometries $w_n \in P$ such that $\sum_n w_n^* w_n = 1$ and $w_n w_n^* = q$ for all n. Define the elements

$$v_{n,k} := (1 \otimes w_n)W_{\varphi_k} p\ .$$

All $v_{n,k}$ belong to $\mathrm{Q}_M(A)$ and $\alpha_{v_{n,k}}$ equals the restriction of β_k to $Ap_{n,k}$ for projections $p_{n,k} \in A$. Since the sum of all $w_n^* w_n$ equals 1, we also have that $\bigvee_n p_{n,k} = p$. Therefore the graphs of the partial automorphisms $\alpha_{v_{n,k}}$ generate an equivalence relation that is isomorphic with $\mathcal{R}$. To conclude the proof, we put $\mathcal{F} := \{v_{n,k} \mid n, k \in \mathbb{N}\}$ and observe that $M = (\mathcal{F} \cup \mathcal{F}^* \cup (A' \cap M))''$. By Lemma 2.1.8, the equivalence relation $\mathcal{R}(A \subset M)$ is also generated by the graphs of the partial automorphisms $\alpha_{v_{n,k}}$. Altogether we have found that $\mathcal{R}(A \subset M)$ and $\mathcal{R}$ are indeed isomorphic. $\square$

2.2 Finite index correspondences

In this section, we introduce the notion of finite index correspondence between two equivalence relations. Roughly speaking, a finite index correspondence is a stable isomorphism up to finite index. Afterwards we state and prove the key technical result needed for the main theorem.

2.2.1 Finite index correspondences between equivalence relations

We start with the definition of a finite index correspondence between two equivalence relations.

Definition 2.2.1. Let $\mathcal{R}$, resp. $\mathcal{R}'$, be a nonsingular countable equivalence relation on a standard probability space (X, μ), resp. (X', μ'). Let $Z \subset X$ and $Z' \subset X'$ be nonnegligible Borel sets that meet almost every orbit and let $\Delta : Z \to Z'$ be a nonsingular isomorphism. Define

$$\mathcal{S}_\Delta := (\mathcal{R}|_Z) \cap (\Delta^{-1} \times \Delta^{-1})(\mathcal{R}'|_{Z'}).$$

We say that Δ is a *finite index correspondence* between $\mathcal{R}$ and $\mathcal{R}'$ if both $\mathcal{S}_\Delta \subset \mathcal{R}|_Z$ and $(\Delta \times \Delta)(\mathcal{S}_\Delta) \subset \mathcal{R}'|_{Z'}$ have finite index. Moreover, if the index of $\mathcal{S}_\Delta$ inside $\mathcal{R}|_Z$ is well defined in the sense of Section 1.4, we call $[\mathcal{R}|_Z : \mathcal{S}_\Delta]$ the *left index of* Δ. Similarly, if the index of $(\Delta \times \Delta)(\mathcal{S}_\Delta)$ inside $\mathcal{R}'|_{Z'}$ is well defined, we call $[\mathcal{R}'|_{Z'} : (\Delta \times \Delta)(\mathcal{S}_\Delta)]$ the *right index of* Δ.

Note that a finite index correspondence with left and right index equal to one, is a stable isomorphism.

The following proposition can be obtained from a more general result of Hamachi and Kosaki ([HK88]). For the convenience of the reader, we include a more direct proof.

Proposition 2.2.2. *Let $\mathcal{S} \subset \mathcal{R}$ be a finite index inclusion of ergodic nonsingular countable equivalence relations on a standard probability space (X, μ). Assume that $\mathcal{R}$ is of type III$_\lambda$, where $0 < \lambda < 1$. Then $\mathcal{S}$ is of type III$_{\lambda^n}$ for some $n \in \mathbb{N}$ with $1 \leq n \leq [\mathcal{R} : \mathcal{S}]$.*

Proof. Denote the Maharam extension (see Section 1.5) of $\mathcal{R}$ by $\widetilde{\mathcal{R}}$ and the Maharam extension of $\mathcal{S}$ by $\widetilde{\mathcal{S}}$. Since $\mathcal{S}$ is a subequivalence relation of $\mathcal{R}$, the Radon-Nikodym 1-cocycle of $\mathcal{S}$ is the restriction of the Radon-Nikodym 1-cocycle of $\mathcal{R}$. This implies that $\widetilde{\mathcal{S}}$ is a subequivalence relation of $\widetilde{\mathcal{R}}$. Also note that for almost every $(x,t) \in X \times \mathbb{R}$ the $\widetilde{\mathcal{R}}$-class of (x,t) consists of $[\mathcal{R} : \mathcal{S}]$ many $\widetilde{\mathcal{S}}$-classes. So we can define the index of the inclusion $\widetilde{\mathcal{S}} \subset \widetilde{\mathcal{R}}$ and moreover $[\widetilde{\mathcal{R}} : \widetilde{\mathcal{S}}] = [\mathcal{R} : \mathcal{S}]$.

Denote by $Y^{\widetilde{\mathcal{R}}}$ the space of ergodic components of $\widetilde{\mathcal{R}}$, i.e. $Y^{\widetilde{\mathcal{R}}}$ satisfies $L^\infty(Y^{\widetilde{\mathcal{R}}}) \cong L^\infty(X \times \mathbb{R})^{\widetilde{\mathcal{R}}}$. Similarly denote by $Y^{\widetilde{\mathcal{S}}}$ the space of ergodic components of $\widetilde{\mathcal{S}}$. We have an ergodic action of $\mathbb{R}$ on $Y^{\widetilde{\mathcal{R}}}$ induced by the ergodic action of $\mathbb{R}$ on $L^\infty(X \times \mathbb{R})^{\widetilde{\mathcal{R}}}$. In the same way, we have an ergodic action of $\mathbb{R}$ on $Y^{\widetilde{\mathcal{S}}}$. From the inclusion $L^\infty(X \times \mathbb{R})^{\widetilde{\mathcal{R}}} \subset L^\infty(X \times \mathbb{R})^{\widetilde{\mathcal{S}}}$ we get an $\mathbb{R}$-equivariant quotient map $f : Y^{\widetilde{\mathcal{S}}} \twoheadrightarrow Y^{\widetilde{\mathcal{R}}}$ satisfying $f^{-1}(Z)$ has measure zero if and only if Z has measure zero.

We show that f is at most $[\mathcal{R} : \mathcal{S}]$-to-one. For that, assume there exists a nonzero integer m and a nonnegligible subset $Z \subset Y^{\widetilde{\mathcal{R}}}$ such that $|f^{-1}(z)| \geq m$ for every $z \in Z$. Then, by Lemma 2.2.3, there exist m nonzero pairwise orthogonal projections inside $L^\infty(X \times \mathbb{R})^{\widetilde{\mathcal{S}}}$ such that their supports in $L^\infty(X \times \mathbb{R})^{\widetilde{\mathcal{R}}}$ all coincide. Equivalently, there exist m nonnegligible disjoint $\widetilde{\mathcal{S}}$-invariant Borel subsets of $X \times \mathcal{R}$ all having the same $\widetilde{\mathcal{R}}$-saturation. This then implies that $m \leq [\widetilde{\mathcal{R}} : \widetilde{\mathcal{S}}] = [\mathcal{R} : \mathcal{S}]$, and hence that f is at most $[\mathcal{R} : \mathcal{S}]$-to-one.

Since $\mathcal{R}$ is of type III_λ, we have that $\mathbb{R} \curvearrowright Y^{\widetilde{\mathcal{R}}}$ is conjugate with $\mathbb{R} \curvearrowright \mathbb{R}/\mathbb{Z}\log(\lambda)$. Hence we have an $\mathbb{R}$-equivariant quotient map $f' : Y^{\widetilde{\mathcal{S}}} \twoheadrightarrow \mathbb{R}/\mathbb{Z}\log(\lambda)$ that is at most $[\mathcal{R} : \mathcal{S}]$-to-one. We also have that $\mathbb{R} \curvearrowright Y^{\widetilde{\mathcal{S}}}$ satisfies exactly one of the following conditions:

1. the action is conjugate with $\mathbb{R} \curvearrowright \mathbb{R}$;
2. the action is conjugate with $\mathbb{R} \curvearrowright \mathbb{R}/\mathbb{Z}\log(\kappa)$ for some $0 < \kappa < 1$;
3. the action is on one point ;
4. the action is properly ergodic.

Since f' is finite-to-one, we see that (1) cannot hold. Furthermore, (3) cannot hold since f' is surjective. That only leaves case (2) and case (4) to be checked.

In case (2), $f'(x) = f'(\log(\kappa)x) = \log(\kappa)f'(x)$ for almost every $x \in Y^{\widetilde{\mathcal{S}}}$. From this we get that $\log(\kappa) \in \mathbb{Z}\log(\lambda)$. Say $\kappa = \lambda^n$. In this way f' can be seen as an $\mathbb{R}$-equivariant map from $\mathbb{R}/\mathbb{Z}n\log(\lambda)$ to $\mathbb{R}/\mathbb{Z}\log(\lambda)$. Necessarily, it is of the form

$$f' : \mathbb{R}/\mathbb{Z}n\log(\lambda) \to \mathbb{R}/\mathbb{Z}\log(\lambda) : t\,\mathbb{Z}n\log(\lambda) \mapsto (t+s)\,\mathbb{Z}\log(\lambda),$$

for some $s \in \mathbb{R}$. Using that f' is at most $[\mathcal{R} : \mathcal{S}]$-to-one, we find that $n \leq [\mathcal{R} : \mathcal{S}]$. So case (2) can only hold when $\kappa = \lambda^n$ for some $n \leq [\mathcal{R} : \mathcal{S}]$.

It remains to prove that (4) cannot hold. Assume by way of reaching a contradiction that the action $\mathbb{R} \curvearrowright Y^{\widetilde{\mathcal{S}}}$ is properly ergodic. We have for almost every $y \in Y^{\widetilde{\mathcal{R}}}$ that $\mathbb{R} \cdot y = Y^{\widetilde{\mathcal{R}}}$, in which case

$$Y^{\widetilde{\mathcal{S}}} = f'^{-1}(Y^{\widetilde{\mathcal{R}}}) = f'^{-1}(\mathbb{R} \cdot y) = \mathbb{R} \cdot f'^{-1}(y).$$

On the other hand, $|f'^{-1}(y)| < +\infty$ for almost every $y \in Y^{\widetilde{\mathcal{R}}}$. So, for almost every $y \in Y^{\widetilde{\mathcal{R}}}$, we have that $\mathbb{R} \cdot f'^{-1}(y)$ is a finite union of $\mathbb{R}$-orbits. Altogether, we see that $Y^{\widetilde{\mathcal{S}}}$ is a finite union of $\mathbb{R}$-orbits. Since $\mathbb{R} \curvearrowright Y^{\widetilde{\mathcal{S}}}$ is properly ergodic, this implies that $Y^{\widetilde{\mathcal{S}}}$ has measure zero. This is a contradiction. □

We used the following lemma in the proof of Proposition 2.2.2.

Lemma 2.2.3. *Let X and Y be standard measure spaces. Let $\Delta : L^\infty(X) \hookrightarrow L^\infty(Y)$ be a normal unital inclusion of von Neumann algebras. Denote by $\Delta_* : Y \twoheadrightarrow X$ the corresponding quotient map, i.e. $\Delta(F) = F \circ \Delta_*$ for every $F \in L^\infty(X)$. Then the following two statements are equivalent:*

- *There exists a nonnegligible subset $Z \subset X$ such that $|\Delta_*^{-1}(z)| \geq n$ for every $z \in Z$;*
- *There exist nonzero pairwise orthogonal projections $p_1, \ldots, p_n \in L^\infty(Y)$ all having the same support in $L^\infty(X)$, i.e. the projections $\mathrm{supp}(E_{L^\infty(X)}(p_i))$ all coincide.*

We are now able to prove the following result.

Proposition 2.2.4. *Let $\mathcal{R}$ and $\mathcal{R}'$ be two ergodic nonsingular countable equivalence relations. Assume that $\mathcal{R}$ is of type III$_\lambda$ and $\mathcal{R}'$ is of type III$_\nu$, where $0 < \lambda, \nu < 1$. Furthermore assume that there exists a finite index correspondence Δ between $\mathcal{R}$ and $\mathcal{R}'$ with left index n_1 and right index n_2. Then $\lambda^{n_1'} = \nu^{n_2'}$ for some $n_1', n_2' \in \mathbb{Z}_{>0}$ with $n_1' \leq n_1$ and $n_2' \leq n_2$.*

Proof. By Lemma 1.5.1, the type of an equivalence relation is preserved under taking stable isomorphisms. Therefore we may assume that both $\mathcal{R}$ and $\mathcal{R}'$ are equivalence relations on the same space (X, μ) such that $[\mathcal{R} : \mathcal{R} \cap \mathcal{R}'] = n_1$ and $[\mathcal{R}' : \mathcal{R} \cap \mathcal{R}'] = n_2$. Write $\mathcal{S}$ for $\mathcal{R} \cap \mathcal{R}'$ and note that $\mathcal{S}$ might not be ergodic. This means that we cannot immediately apply Proposition 2.2.2 to these inclusions. Let us work around this.

Consider the nonsingular quotient map $f : X^{\mathcal{S}} \twoheadrightarrow X^{\mathcal{R}} = \{*\}$ coming from the inclusion $L^\infty(X)^{\mathcal{R}} \subset L^\infty(X)^{\mathcal{S}}$. As before, we have that f is finite-to-one. Hence there exists a point in $X^{\mathcal{S}}$ having nonzero measure. On the other hand, every point in $X^{\mathcal{S}}$ corresponds with an $\mathcal{S}$-invariant Borel set on which $\mathcal{S}$ is ergodic. Therefore we get that there exists at least one nonnegligible $\mathcal{S}$-invariant Borel set X' on which $\mathcal{S}$ is ergodic.

Now note that $\mathcal{R}|_{X'}$ is of type III_λ, $\mathcal{R}'|_{X'}$ is of type III_ν, $[\mathcal{R}|_{X'} : \mathcal{S}|_{X'}] \leq n_1$ and $[\mathcal{R}'|_{X'} : \mathcal{S}|_{X'}] \leq n_2$. Applying Proposition 2.2.2, we get that $\mathcal{S}|_{X'}$ is of type $\mathrm{III}_{\lambda^{n_1'}}$ and of type $\mathrm{III}_{\nu^{n_2'}}$ for some $n_1', n_2' \in \mathbb{Z}_{>0}$ with $n_1' \leq n_1$ and $n_2' \leq n_2$. This concludes the proof. $\square$

We end this subsection with an example of a finite index correspondence for which the bounds in Proposition 2.2.4 are attained.

Example 2.2.5. Let $n, m \in \mathbb{Z}$ be nonzero integers satisfying $|n| < |m|$. Denote by k the greatest common divisor of $|n|$ and $|m|$. Put $n = kn_0$ and $m = km_0$. Furthermore let $H_{n,m}$ denote the $\mathbb{Z}$-linear span of $\{\frac{1}{kn_0^s m_0^t} \mid s, t \in \mathbb{N}, s + t > 0\}$ and define the action $\mathbb{Z} \curvearrowright^{\alpha_{n,m}} H_{n,m}$ as

$$z \cdot x = \left(\frac{n}{m}\right)^z x \text{ for every } z \in \mathbb{Z} \text{ and } x \in H_{n,m}.$$

We write $G_{n,m} := H_{n,m} \rtimes_{\alpha_{n,m}} \mathbb{Z}$. Then $G_{n,m}$ acts on $\mathbb{R}$ by affine transformations in the following way:

$$\left(\frac{z_1}{kn_0^s m_0^t}, z_2\right) \cdot x = \left(\frac{n}{m}\right)^{z_2} x + \frac{z_1}{kn_0^s m_0^t} \text{ for every } x \in \mathbb{R}.$$

Define $\mathcal{R} := \mathcal{R}(G_{n,m} \curvearrowright \mathbb{R})$. With $\varphi : \mathbb{R} \to \mathbb{T} : x \mapsto e^{2\pi i x}$, we have that $\mathcal{R}_{n,m} := (\varphi \times \varphi)(\mathcal{R}) = (\varphi \times \varphi)(\mathcal{R}|_{[0,1]})$ is a nonsingular countable Borel equivalence relation on the circle $\mathbb{T}$.

Now define the set $\mathcal{R}_0 := \{(y, z) \in \mathbb{T} \times \mathbb{T} \mid y^m = z^n\}$. Let us show that $\mathcal{R}_{n,m}$ is the smallest equivalence relation on $\mathbb{T}$ containing $\mathcal{R}_0$. For that, fix numbers $s, t \in \mathbb{N}$ such that $s + t > 0$. Since n_0 and m_0 are coprime, there exist $p, q \in \mathbb{Z}$ satisfying $pn_0^{s+t-1} + qm_0^{s+t-1} = 1$. A direct computation yields

$$\left(\frac{1}{kn_0^s m_0^t}, 0\right) = (0,1)^{t-1} \left(\frac{p}{km_0}, 1\right) (0,1)^{-t-s} \left(\frac{q}{km_0}, 1\right) (0,1)^{s-1}.$$

And since

$$\left(\frac{z_1}{kn_0^s m_0^t}, z_2\right) = \left(\frac{1}{kn_0^s m_0^t}, 0\right)^{z_1} (0,1)^{z_2},$$

we have that $G_{n,m}$ is generated as a group by the set $G_0 := \{(\frac{z}{km_0}, 1) \mid z \in \mathbb{Z}\}$. Together with the fact that $\mathcal{R}_0 = (\varphi \times \varphi)(\{(g \cdot x, x) \mid g \in G_0\})$, we see that $\mathcal{R}_{n,m}$ is indeed the smallest equivalence relation containing $\mathcal{R}_0$.

As in Section 1.5, denote by $\omega : \mathcal{R}_{n,m} \to \mathbb{R}$ the Radon-Nikodym 1-cocycle. Denote by $\Lambda \subset \mathbb{T}$ the subgroup given by

$$\Lambda := \Big\{\exp\Big(\frac{2\pi i s}{(n_0 m_0)^t}\Big) \Bigm| s \in \mathbb{Z}, t \in \mathbb{N}\Big\} .$$

For every $z \in \Lambda$, we denote by $\alpha_z : \mathbb{T} \to \mathbb{T}$ the rotation $\alpha_z(y) = zy$. We have $\operatorname{graph} \alpha_z \subset \mathcal{R}_{n,m}$ for all $z \in \Lambda$. Since all α_z are measure preserving, we actually have $\operatorname{graph} \alpha_z \subset \operatorname{Ker}(\omega)$. Since $\Lambda \subset \mathbb{T}$ is a dense subgroup, it follows that $\operatorname{Ker}(\omega)$ is an ergodic equivalence relation. In particular, $\mathcal{R}_{n,m}$ is ergodic. A direct computation shows that $\omega(y, z) = \log(|n|/|m|)$ for all $(y, z) \in \mathcal{R}_0$. Since $\mathcal{R}_0$ generates the equivalence relation $\mathcal{R}_{n,m}$, it follows that the essential image of ω equals $\log(|n|/|m|)\mathbb{Z}$. Using Lemma 1.5.2, we conclude that $\mathcal{R}_{n,m}$ is of type $\mathrm{III}_{|n|/|m|}$. In fact $\mathcal{R}_{n,m}$ is the unique hyperfinite ergodic countable equivalence relation of type $\mathrm{III}_{|n|/|m|}$. Indeed, by Theorem 1.4.8, we have that $\mathcal{R}$ is hyperfinite. Therefore the restriction $\mathcal{R}|_{[0,1]}$ is hyperfinite. Since $\mathcal{R}_{n,m}$ is isomorphic with this equivalence relation, we have that $\mathcal{R}_{n,m}$ is also hyperfinite. Altogether, $\mathcal{R}_{n,m}$ is indeed the unique hyperfinite ergodic countable equivalence relation of type $\mathrm{III}_{|n|/|m|}$.

We can now begin with the actual example. Let $n, m, p, q \in \mathbb{Z}$ satisfy the following three conditions:

- $1 \leq n < m$ and $1 \leq p < q$;
- $\gcd(n, m) = 1$ and $\gcd(p, q) = 1$;
- there exists a nonzero positive integer l with $(n/m)^l \in (p/q)^{\mathbb{Z}}$.

Let s be the smallest nonzero positive integer such that $(n/m)^s \in (p/q)^{\mathbb{Z}}$ and let $t \in \mathbb{Z}_{>0}$ satisfy $(n/m)^s = (p/q)^t$. We claim that $\mathcal{R}_{n,m} \cap \mathcal{R}_{p,q}$ has index s in $\mathcal{R}_{n,m}$ and index t in $\mathcal{R}_{p,q}$. In other words, we claim that the identity function on $\mathbb{T}$ is a finite index correspondence between $\mathcal{R}_{n,m}$ and $\mathcal{R}_{p,q}$ with left index s and right index t.

Let us first prove that $\mathcal{R}_{n,m} \cap \mathcal{R}_{p,q} = \mathcal{R}_{n^s,m^s} = \mathcal{R}_{p^t,q^t}$. Since $\gcd(n, m) = 1$ and $\gcd(p, q) = 1$, we have that $n^s = p^t$ and $m^s = q^t$. This already shows that $\mathcal{R}_{n^s,m^s} = \mathcal{R}_{p^t,q^t}$. Since $\mathcal{R}_{n^s,m^s}$ is a subequivalence relation of $\mathcal{R}_{n,m}$ and $\mathcal{R}_{p^t,q^t}$ is a subequivalence relation of $\mathcal{R}_{p,q}$, we moreover have that $\mathcal{R}_{n^s,m^s} = \mathcal{R}_{p^t,q^t} \subset \mathcal{R}_{n,m} \cap \mathcal{R}_{p,q}$. It remains to prove the converse inclusion. For that, let ψ be an element of the full pseudogroup of $\mathcal{R}_{n,m} \cap \mathcal{R}_{p,q}$. We need to prove that ψ

belongs to the full pseudogroup of $\mathcal{R}_{n^s,m^s}$. To that end let φ_0 be the restriction of φ to the unit interval $[0,1]$. Every element of the full pseudogroup of $\mathcal{R}_{n,m}$ is a glueing of elements of the form $\varphi_0 \circ \alpha_{n,m}(g) \circ \varphi_0^{-1}|_{\mathcal{U}}$, where $g \in G_{n,m}$ and $\mathcal{U} \subset \mathbb{T}$ is a Borel subset satisfying $(\alpha_{n,m}(g) \circ \varphi_0^{-1})(\mathcal{U}) \subset [0,1]$. So to show that ψ belongs to the full pseudogroup of $\mathcal{R}_{n^s,m^s}$, it suffices to assume that ψ is of the form $\psi = \varphi_0 \circ \alpha_{n,m}(g) \circ \varphi_0^{-1}|_U$. Since ψ belongs to the full pseudogroup of $\mathcal{R}_{n,m} \cap \mathcal{R}_{p,q}$, we have that

$$\omega(x, \psi(x)) \in \mathbb{Z}\log(n/m) \cap \mathbb{Z}\log(p/q) = \mathbb{Z}\log((n/m)^s),$$

for almost every $x \in \mathcal{U}$. Hence $\omega(x, \alpha_{n,m}(g)(x)) \in \mathbb{Z}\log((n/m)^s)$ for almost every $x \in \varphi_0^{-1}(\mathcal{U})$. This means that g is an element of $H_{n,m} \rtimes_{\alpha_{n,m}} s\mathbb{Z} = G_{n^s,m^s}$. Altogether we see that the full pseudogroup of $\mathcal{R}_{n,m} \cap \mathcal{R}_{p,q}$ is indeed contained in the full pseudogroup of $\mathcal{R}_{n^s,m^s}$.

To end this example we still need to show that $\mathcal{R}_{n^s,m^s}$ has index s in $\mathcal{R}_{n,m}$ and that $\mathcal{R}_{p^t,q^t}$ has index t in $\mathcal{R}_{p,q}$. Let us only show the first statement, since the second statement can be shown in completely the same way. First of all, we have that $s \leq [\mathcal{R}_{n,m} : \mathcal{R}_{n^s,m^s}]$ by Proposition 2.2.2. On the other hand, $[G_{n,m} : G_{n^s,m^s}] = s$. So $\mathcal{R}(G_{n^s,m^s} \curvearrowright \mathbb{R})$ is at most index s inside $\mathcal{R}(G_{n,m} \curvearrowright \mathbb{R})$. Restricting both relations to $[0,1]$ can only reduce the index even more. Therefore $\mathcal{R}_{n^s,m^s}$ is at most index s inside $\mathcal{R}_{n,m}$. Altogether $[\mathcal{R}_{n,m} : \mathcal{R}_{n^s,m^s}] = s$, as we wanted to show.

2.2.2 From finite index bimodules to finite index correspondences

Let (X,μ) be a standard probability space. Throughout this subsection, we use the following notation. For every nonsingular partial automorphism φ of (X,μ), we denote by α_φ the corresponding partial automorphism of $L^\infty(X,\mu)$. Similarly, for every partial automorphism α of $L^\infty(X,\mu)$, we denote by φ_α the corresponding nonsingular partial automorphism of (X,μ).

The following theorem is a generalization of Theorem 3.3 from [MV13] and is the key technical result needed for Theorem A.

Theorem 2.2.6. *Let (M,τ_M) and (N,τ_N) be II_1 factors having separable L^2-spaces. Let $A \subset M$ and $B \subset N$ be abelian quasi-regular von Neumann subalgebras satisfying $\mathcal{Z}(A' \cap M) = A$ and $\mathcal{Z}(B' \cap N) = B$. Let ${}_M\mathcal{H}_N$ be a finite index M-N-bimodule. Write $x := dim_{M\text{-}}(\mathcal{H})$ and $y := dim_{\text{-}N}(\mathcal{H})$. Assume that $\mathcal{H}$ contains a nonzero bifinite A-B-subbimodule ${}_A\mathcal{K}_B$. Then there exists a finite index correspondence between $\mathcal{R}(A \subset M)$ and $\mathcal{R}(B \subset N)$ with both left and right index smaller than xy.*

Proof. By Lemma 2.1.2, we may assume that ${}_A\mathcal{K}_B \cong_\alpha {}_A\mathcal{H}(\beta)_B$ where $\beta : Aq \to Bp$ is a partial isomorphism. Denote the vector $\alpha^{-1}(p) \in \mathcal{K}$ by ξ, then

- $a\xi = aq\xi = \xi\beta(aq)$ for every $a \in A$;
- ξ generates $\mathcal{K}$ as a left A-module and as a right B-module;
- ξ is separating for the left Aq-action and the right Bp-action.

Choose standard probability spaces (X, μ) and (Y, ν) such that $A = L^\infty(X, \mu)$ and $B = L^\infty(Y, \nu)$. Write $q = \chi_{X'}$ and $p = \chi_{Y'}$. Let Δ be the partial isomorphism from X' onto Y', corresponding with β. We prove that Δ is a finite index correspondence between $\mathcal{R}(A \subset M)$ and $\mathcal{R}(B \subset N)$ with left and right index smaller than xy. To that end, define

$$\mathcal{S} = \mathcal{R}(A \subset M)|_{X'} \ \cap \ (\Delta^{-1}{\times}\Delta^{-1})(\mathcal{R}(B \subset N)|_{Y'}).$$

We must show that

$$[\mathcal{R}(A \subset M)|_{X'} : \mathcal{S}] \leq xy \tag{2.4}$$

and

$$[\mathcal{R}(B \subset N)|_{Y'} : (\Delta \times \Delta)(\mathcal{S})] \leq xy. \tag{2.5}$$

It is clear that, by symmetry, we only need to prove inequality (2.4). To prove this we argue by contradiction. So assume that inequality (2.4) does not hold. Using Lemma 1.4.9 there exists an integer number $n > xy$, a nonzero projection $q_0 = \chi_{X_0}$ in Aq and elements $v_1, \ldots, v_n \in \mathrm{Q}_M(A)$ satisfying

- $\mathrm{supp}(E_A(v_i v_i^*)) = q_0$, $\mathrm{supp}(E_A(v_i^* v_i)) \leq q$ for every $i \in \{1, \ldots, n\}$;
- $[\varphi_{\alpha_{v_1}}(x)]_{\mathcal{S}}, \ldots, [\varphi_{\alpha_{v_n}}(x)]_{\mathcal{S}}$ are disjoint for almost every $x \in X_0$.

Denote by $\mathcal{H}^{(i)}$ the $(A' \cap M)$-N-subbimodule $\overline{\mathrm{span}}(A' \cap M)v_i(A' \cap M)\xi N$ of $\mathcal{H}$. We shall arrive at the contradiction that $n \leq xy$ by examining the right dimensions of these bimodules. We start with the following claim.

Claim 1. The bimodules $\mathcal{H}^{(i)}$ and $\mathcal{H}^{(j)}$ are orthogonal whenever $i \neq j$.

Let $i, j \in \{1, \ldots, n\}$ such that $i \neq j$. By Lemma 2.1.3, $Q_N(B)'' = QN_N(B)'' = N$. We must therefore show that every element of

$$(A' \cap M)v_j^*(A' \cap M)v_i(A' \cap M)\xi Q_N(B)$$

is orthogonal to $\mathcal{K}$. So let $a_1, a_2, a_3 \in A' \cap M$ and $w \in Q_N(B)$. Write $v := a_1 v_j^* a_2 v_i a_3$. We need to prove that $v\xi w$ is orthogonal to $\mathcal{K}$. First note that

$v \in \mathrm{Q}_M(A)$ and that α_v is a restriction of $\alpha_{v_i} \circ \alpha_{v_j}^{-1}$. Hence no nonnegligible restriction of φ_{α_v} is an element of $[[\mathcal{S}]]$ or equivalently no nonnegligible restriction of φ_{α_v} is an element of $[[(\Delta^{-1}\times\Delta^{-1})(\mathcal{R}(B \subset N)|_{Y'})]]$.

Define $P : \mathcal{H} \to \mathcal{K}$ to be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$. We must show that $P(v\xi w) = 0$. Assume, by way of reaching a contradiction, that $P(v\xi w) \neq 0$. Let $q_1 \in Aq$ be the smallest projection q' in A such that $q'P(v\xi w) = P(v\xi w)$. Since $\mathrm{dom}(\alpha_w \circ \beta \circ \alpha_v) = A\alpha_v^{-1}(p_v\beta^{-1}(pq_w))$ and $\alpha_v^{-1}(p_v\beta^{-1}(pq_w))P(v\xi w) = P(v\xi w)$, we see that $Aq_1 \subset \mathrm{dom}(\alpha_w \circ \beta \circ \alpha_v)$. Furthermore, since $P(v\xi w) \in \mathcal{K}$, we have that $aP(v\xi w) = P(v\xi w)\beta(aq_1)$, for every $a \in A$. Lastly, we also have that

$$\begin{aligned} aP(v\xi w) &= P(v\alpha_v(aq_1)\xi w) \\ &= P(v\xi\beta(\alpha_v(aq_1))w) \\ &= P(v\xi w)\alpha_w(\beta(\alpha_v(aq_1))), \end{aligned}$$

for every $a \in A$. Therefore $P(v\xi w)\beta(aq_1) = P(v\xi w)\alpha_w(\beta(\alpha_v(aq_1)))$ for every $a \in A$. But then $\alpha_w \circ \beta \circ \alpha_v|_{Aq_1} = \beta|_{Aq_1}$ and hence $\alpha_v|_{Aq_1} = \beta^{-1} \circ \alpha_w^{-1} \circ \beta|_{Aq_1}$. So we have found that there does exist some nonnegligible restriction of φ_{α_v} that is inside $[[(\Delta^{-1}\times\Delta^{-1})(\mathcal{R}(B \subset N)|_{Y'})]]$. This is a contradiction and hence we have that $P(v\xi w) = 0$. This ends the proof of the claim.

Let us continue with the proof of the theorem. Since ${}_M\mathcal{H}_N$ is a finite index M-N-bimodule with $dim_{M\text{-}}(\mathcal{H}) = x$ and $dim_{\text{-}N}(\mathcal{H}) = y$, we find

- a projection $r \in N^\infty$ with $\mathrm{Tr} \otimes \tau_N(r) = y$,
- a normal $*$-homomorphism $\psi : M \to rN^\infty r$ with $[rN^\infty r : \psi(M)] = xy$,

such that ${}_M\mathcal{H}_N \cong {}_{\psi(M)}r(l^2(\mathbb{N}) \otimes L^2(N))_N$. We have the following claim.

Claim 2. The N-dimension of $\mathcal{H}^{(i)}$ is at least $(\mathrm{Tr} \otimes \tau_N)(\psi(q_0))/(xy)$.

We have that $\mathcal{H}^{(i)}$ is an $(A' \cap M)$-N-subbimodule of $\mathcal{H}$. Therefore

$${}_{(A' \cap M)}\mathcal{H}^{(i)}{}_N \cong {}_{\psi(A' \cap M)}s(l^2(\mathbb{N}) \otimes L^2(N))_N,$$

for some projection $s \in \psi(A' \cap M)' \cap rN^\infty r$. By definition we have that

$$dim_{\text{-}N}(\mathcal{H}^{(i)}) = (\mathrm{Tr} \otimes \tau_N)(s).$$

Applying Lemma 2.2.7 below to the inclusion $\psi(M) \subset rN^\infty r$ and the projection $s \in rN^\infty r$, we find that $(\mathrm{Tr} \otimes \tau_N)(s) \geq (\mathrm{Tr} \otimes \tau_N)(\mathrm{supp}(E_{\psi(M)}(s)))/(xy)$. Since the projection s is an element of $\psi(A' \cap M)' \cap rN^\infty r$, we also have that

$E_{\psi(M)}(s) \in \psi(A' \cap M)' \cap \psi(M) = \psi(A)$. Therefore $E_{\psi(M)}(s) = E_{\psi(A)}(s)$, and hence

$$(\mathrm{Tr} \otimes \tau_N)(s) \geq (\mathrm{Tr} \otimes \tau_N)(\mathrm{supp}(E_{\psi(A)}(s)))/(xy).$$

To prove our claim it therefore suffices to show that $\mathrm{supp}(E_{\psi(A)}(s)) = \psi(q_0)$.

In order to do this, we start with the following observation. Let x be an arbitrary element of M. Note that the join of the left supports of all elements in $(A' \cap M)x$ is equal to the orthogonal projection of $L^2(M)$ onto $\overline{\mathrm{span}}(A' \cap M)xM$. Indeed, the join of the left supports of all elements in $(A' \cap M)x$ is the smallest projection p_x satisfying $p_x\eta = \eta$ for every $\eta \in (A' \cap M)xL^2(M)$. This then immediately yields the equality of the two projections. On the other hand, by Lemma 2.1.4, we have that the orthogonal projection of $L^2(M)$ onto $\overline{\mathrm{span}}(A' \cap M)xM$ is equal to $\mathrm{supp}(E_A(xx^*))$. Hence the join of the left supports of all elements in $(A' \cap M)x$ is equal to $\mathrm{supp}(E_A(xx^*))$. Or equivalently, the join of the right supports of all elements in $x(A' \cap M)$ is equal to $\mathrm{supp}(E_A(x^*x))$.

Let us go back to showing that $\mathrm{supp}(E_{\psi(A)}(s)) = \psi(q_0)$. Note that $q_0\mathcal{H}^{(i)} = \mathcal{H}^{(i)}$. Let q_2 be the smallest projection q' in A such that $q'\mathcal{H}^{(i)} = \mathcal{H}^{(i)}$ and define $q_3 = q_0 - q_2$. Then $q_3 \leq q_0$ and $q_3 v_i (A' \cap M)\xi = 0$. By our observation, we find that $\mathrm{supp}(E_A(v_i^* q_3 v_i))\xi = 0$. Since $q_3 \leq q_0$, we have that $\mathrm{supp}(E_A(v_i^* q_3 v_i)) = \alpha_{v_i}(q_3)$. So $\alpha_{v_i}(q_3)\xi = 0$. Since ξ is separating for the left Aq-action, we find that $\alpha_{v_i}(q_3) = 0$ or equivalently that $q_3 = 0$. In other words, we have found that q_0 is the smallest projection q' in A such that $q'\mathcal{H}^{(i)} = \mathcal{H}^{(i)}$. Since ${}_A\mathcal{H}^{(i)} \cong {}_{\psi(A)}s(l^2(\mathbb{N}) \otimes L^2(N))$, we see that this implies that $\psi(q_0)$ is the smallest projection q' in $\psi(A)$ such that $q's = s$. Hence $\psi(q_0)$ must be equal to $\mathrm{supp}(E_{\psi(A)}(s))$ by Lemma 2.1.1. This ends the proof of Claim 2.

We can now finish the proof of the theorem. Using Claim 1 and Claim 2, we have that

$$(Tr \otimes \tau_N)(\psi(q_0)) = \dim_{\text{-}N} q_0\mathcal{H} \geq \sum_{i=1}^{n} \dim_{\text{-}N}(\mathcal{H}^{(i)}) \geq n\frac{(Tr \otimes \tau_N)(\psi(p_0))}{xy}.$$

Therefore $n \leq xy$ and we have arrived at the desired contradiction. This ends the proof of the theorem. $\square$

In the proof of Theorem 2.2.6, we used the following lemma.

Lemma 2.2.7. *Let (M, τ) be a II$_1$ factor with a finite index von Neumann subalgebra N. Let $p \in M$ be a projection and write $p_N = \mathrm{supp}(E_N(p))$. Then $\tau(p) \geq \frac{\tau(p_N)}{[M:N]}$.*

Proof. Let $M_1 = \langle M, e_N \rangle$ be the basic construction of $N \subset M$. Define $x = e_N p \in M_1$. Then $xx^* = E_N(p)e_N$ and so $\mathrm{supp}(xx^*) = p_N e_N$. But then

$\tau(p_N) = \mathrm{Tr}(p_N e_N) = \mathrm{Tr}(\mathrm{supp}(xx^*)) = \mathrm{Tr}(\mathrm{supp}(x^*x))$. Since $x^*x = pe_Np$, we have that $\mathrm{supp}(x^*x) \leq p$, so $\tau(p_N) \leq \mathrm{Tr}(p)$. Finally since M is a II_1 factor and $N \subset M$ is a finite index inclusion, we have that $\mathrm{Tr}\,|_M(\cdot) = [M : N]\tau(\cdot)$. This yields the desired inequality $\tau(p_N) \leq [M : N]\tau(p)$. □

2.3 Proof of the main theorem

We are ready to prove Theorem A. So let k and l be nonzero positive integers. For every $i \in \{1, \dots, k\}$ and every $j \in \{1, \dots, l\}$, let $n_i, m_i, p_j, q_j \in \mathbb{Z}$ satisfy $2 \leq n_i < |m_i|$ and $2 \leq p_j < |q_j|$. We make use the following notation.

- For all $1 \leq i \leq k$, put $\Gamma_i := \mathrm{BS}(n_i, m_i) = \langle a_i, b_i \mid b_i a_i^{n_i} b_i^{-1} = a_i^{m_i} \rangle$.
- For all $1 \leq j \leq l$, put $\Lambda_j := \mathrm{BS}(p_j, q_j) = \langle c_j, d_j \mid d_j c_j^{p_j} d_j^{-1} = c_j^{q_j} \rangle$.
- Put $\Gamma := \prod_{i=1}^k \Gamma_i$ and $\Lambda := \prod_{j=1}^l \Lambda_j$.

Assume that $\mathcal{H}$ is a nonzero finite index $L(\Gamma)$-$L(\Lambda)$-bimodule. We need to prove that $k = l$ and that there exists a permutation $\sigma \in \mathrm{Sym}(k)$ such that for every $i \in \{1, \dots, k\}$,

$$(n_i/|m_i|)^{r_i} = (p_{\sigma(i)}/|q_{\sigma(i)}|)^{s_i}$$

for some $r_i, s_i \in \mathbb{Z}$ with $1 \leq r_i, s_i \leq \dim_{M-}(\mathcal{H}) \dim_{-N}(\mathcal{H})$. We split the proof of Theorem A into two steps.

In the first step we examine the positions of the subalgebras $L(\hat{\Gamma}_i)$ in $L(\Gamma)$ and the positions of the subalgebras $L(\hat{\Lambda}_j)$ in $L(\Lambda)$, where

- for all $1 \leq i \leq k$, $\hat{\Gamma}_i := \Gamma_1 \times \ldots \times \Gamma_{i-1} \times \langle a_i \rangle \times \Gamma_{i+1} \times \ldots \times \Gamma_k$;
- for all $1 \leq j \leq l$, $\hat{\Lambda}_j := \Lambda_1 \times \ldots \times \Lambda_{j-1} \times \langle c_j \rangle \times \Lambda_{j+1} \times \ldots \times \Lambda_l$.

We show that the subalgebras $L(\hat{\Gamma}_i)$, respectively $L(\hat{\Lambda}_j)$, have up to intertwining, canonical positions in $L(\Gamma)$, respectively $L(\Lambda)$. This will allow us to deduce that $k = l$ and that there exists a permutation $\sigma \in \mathrm{Sym}(k)$ such that the bimodule $\mathcal{H}$ contains a nonzero finite index $\mathcal{Z}(L(\hat{\Gamma}_i))$-$\mathcal{Z}(L(\hat{\Lambda}_{\sigma(i)}))$-subbimodule for every $1 \leq i \leq k$.

In the second step, we use our key technical result to find that there exists a finite index correspondence between the relations $\mathcal{R}(\mathcal{Z}(L(\hat{\Gamma}_i)) \subset L(\Gamma))$ and $\mathcal{R}(\mathcal{Z}(L(\hat{\Lambda}_{\sigma(i)})) \subset L(\Lambda))$. By looking at the types of both relations we will be able to deduce that for every $i \in \{1, \dots, k\}$, we have $(n_i/|m_i|)^{r_i} = (p_{\sigma(i)}/|q_{\sigma(i)}|)^{s_i}$ for some $r_i, s_i \in \mathbb{Z}$ with $1 \leq r_i, s_i \leq \dim_{L(\Gamma)-}(\mathcal{H}) \dim_{-L(\Lambda)}(\mathcal{H})$, thus ending the proof.

2.3.1 First step of the proof

We start off with a technical result on relative amenability. For it, we need the notion of commuting square. Two von Neumann subalgebras Q_1 and Q_2 of a tracial von Neumann algebra (M, τ) form a *commuting square* if $E_{Q_1} \circ E_{Q_2} = E_{Q_2} \circ E_{Q_1}$ (see e.g. [JS97, Definition 5.1.7]). Note that this property is equivalent to the following two conditions:

- $E_{Q_1}(x) \in Q_1 \cap Q_2$ for every $x \in Q_2$;
- $E_{Q_2}(x) \in Q_1 \cap Q_2$ for every $x \in Q_1$.

An easy example of a commuting square is the following. Let $H_1, H_2 \leq G$ be countable discrete groups and let n be a nonzero positive integer. Then $L(H_1)^n$ and $L(H_2)^n$ form a commuting square inside $L(G)^n$. Indeed, this follows immediately from the fact that $e_{l^2(H_1)} \circ e_{l^2(H_2)} = e_{l^2(H_1 \cap H_2)} = e_{l^2(H_2)} \circ e_{l^2(H_1)}$.

The following result generalizes Proposition 2.7 from [PV11].

Proposition 2.3.1. *Let (M, τ) be a tracial von Neumann algebra with von Neumann subalgebras $Q_1, Q_2 \subset M$. Assume that there exists a countable subgroup $\mathcal{G} \subset \mathcal{U}(M)$ satisfying the following three conditions.*

1. *$\mathcal{G}'' = M$.*
2. *For every $u \in \mathcal{G}$, uQ_1u^* and Q_2 form a commuting square inside M.*
3. *For every $u \in \mathcal{G}$, $Q_1 \cap uQ_1u^* \cap Q_2$ is co-amenable in $uQ_1u^* \cap Q_2$.*

Let p be a nonzero projection of M. If a von Neumann algebra $P \subset pMp$ is amenable relative to both Q_1 and Q_2, then P is amenable relative to $Q_1 \cap Q_2$.

Proof. Define $\mathcal{H} := L^2(M) \otimes_{Q_1} L^2(M) \otimes_{Q_2} L^2(M)$. The first part of the proof of Proposition 2.7 from [PV11] gives us that ${}_{pMp}L^2(pMp)_P$ is weakly contained in ${}_{pMp}(p\mathcal{H}p)_P$.

Whenever $u \in \mathcal{G}$, denote by $\mathcal{H}_u \subset \mathcal{H}$ the closed linear span of the vectors $\{x \otimes_{Q_1} u \otimes_{Q_2} y \mid x, y \in M\}$. Note that $\mathcal{H}_u$ is an M-M-subbimodule of $\mathcal{H}$. Furthermore, since $\overline{\text{span}}^{||\cdot||_2}\mathcal{G} = L^2(M)$, we have that the bimodules $\mathcal{H}_u$

together span a dense subspace of $\mathcal{H}$. Denote for every $u \in \mathcal{G}$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_u$ by p_u and write $\mathcal{G} = \{u_1, u_2, \ldots\}$. We define $T \in \mathrm{B}({}_M\mathcal{H}_M, {}_M(\bigoplus_n \mathcal{H}_{u_n})_M)$ as

$$T(\xi) = \bigoplus_n \frac{1}{2^n} p_{u_n}(\xi).$$

Taking the polar decomposition $T = V|T|$, we get an M-M-bimodular partial isometry V from $\mathcal{H}$ to $\bigoplus_{u\in\mathcal{G}} \mathcal{H}_u$. Since T is injective, also V is injective and hence is an isometry. Altogether, we have found that there exists an M-M-bimodular isometry from $\mathcal{H}$ to $\bigoplus_{u\in\mathcal{G}} \mathcal{H}_u$.

Whenever $u \in \mathcal{G}$, write Q_u for $u^*Q_1u \cap Q_2$. The commuting square condition guarantees that the formula

$$x \otimes_{Q_1} u \otimes_{Q_2} y \mapsto xu \otimes_{Q_u} y$$

defines an M-M-bimodular unitary from $\mathcal{H}_u$ onto $L^2(M) \otimes_{Q_u} L^2(M)$. It follows that there exists an M-M-bimodular isometry from $\mathcal{H}$ to the direct sum $\bigoplus_{u\in\mathcal{G}}(L^2(M) \otimes_{Q_u} L^2(M))$. Since ${}_{pMp}L^2(pMp)_P$ is weakly contained in ${}_{pMp}(p\mathcal{H}p)_P$, we therefore get that

$$ {}_{pMp}L^2(pMp)_P \prec {}_{pMp}(\bigoplus_{u\in\mathcal{G}} pL^2(M) \otimes_{Q_u} L^2(M)p)_P. \tag{2.6}$$

Since by assumption $Q_1 \cap Q_u$ is a co-amenable subalgebra of Q_u, we have that

$$ {}_{Q_u}L^2(Q_u)_{Q_u} \prec {}_{Q_u}L^2(Q_u) \otimes_{Q_1\cap Q_u} L^2(Q_u)_{Q_u}.$$

Taking the Connes tensor product with ${}_{pMp}pL^2(M)_{Q_u}$ on the left and with ${}_{Q_u}L^2(M)p_{pMp}$ on the right yields

$$ {}_{pMp}(pL^2(M) \otimes_{Q_u} L^2(M)p)_{pMp} \prec {}_{pMp}(pL^2(M) \otimes_{Q_1\cap Q_u} L^2(M)p)_{pMp},$$

by Lemma 1.7.5. Combined with (2.6), this results in

$$ {}_{pMp}L^2(pMp)_P \prec {}_{pMp}(\bigoplus_{u\in\mathcal{G}}(pL^2(M) \otimes_{Q_1\cap Q_u} L^2(M)p))_P.$$

This allows us then to take a net of unit vectors (ξ_n) in an infinite multiple of $\bigoplus_{u\in\mathcal{G}}(pL^2(M) \otimes_{Q_1\cap Q_u} L^2(M)p)$ satisfying $\langle x\xi_n, \xi_n\rangle \to \tau_{pMp}(x)$ for every $x \in pMp$ and $||y\xi_n - \xi_n y|| \to 0$ for every $y \in P$.

Since $\langle M, e_{Q_1\cap Q_2}\rangle$ is a subalgebra of $\langle M, e_{Q_1\cap Q_u}\rangle$, the formula

$$S(\xi \otimes_{Q_1\cap Q_u} \eta) = S\xi \otimes_{Q_1\cap Q_u} \eta$$

provides a normal representation of $p\langle M, e_{Q_1\cap Q_2}\rangle p$ on the direct sum $\bigoplus_{u\in\mathcal{G}}(pL^2(M)\otimes_{Q_1\cap Q_u} L^2(M)p)$ that commutes with the right P-action. Then choosing a weak* limit point of the net of states $S\mapsto\langle S\xi_n,\xi_n\rangle$, we have found a P-central state on $p\langle M, e_{Q_1\cap Q_2}\rangle p$ whose restriction to pMp is τ_{pMp}. So $P\subset pMp$ is amenable relative to $Q_1\cap Q_2$. $\square$

We have the following corollary of Proposition 2.3.1.

Corollary 2.3.2. *Let (M,τ) be a tracial von Neumann algebra with von Neumann subalgebras $Q_1, Q_2\subset M$. Assume that there exists a countable subgroup $\mathcal{G}\subset\mathcal{U}(M)$ satisfying the following three conditions.*

1. *$\mathcal{G}'' = M$.*
2. *For every $u\in\mathcal{G}$, uQ_1u^* and Q_2 form a commuting square inside M.*
3. *For every $u\in\mathcal{G}$, $Q_1\cap uQ_1u^*$ is a finite index subalgebra of Q_1.*

Let p be a nonzero projection of M. If a von Neumann algebra $P\subset pMp$ is amenable relative to both Q_1 and Q_2, then P is amenable relative to $Q_1\cap Q_2$.

Proof. Using that $Q_1\cap uQ_1u^*$ is a finite index subalgebra of Q_1 and combining this with the fact that uQ_1u^* and Q_2 form a commuting square inside M, we show that $Q_1\cap uQ_1u^*\cap Q_2$ is a co-amenable subalgebra of $uQ_1u^*\cap Q_2$. By Proposition 2.3.1, this will end the proof.

We begin with the following observation. Fix $u\in\mathcal{G}$ and let $x\in Q_1\cap uQ_1u^*$. Then $E_{Q_2}(x)\in Q_1\cap Q_2$ since Q_1 and Q_2 form a commuting square. On the other hand, $E_{Q_2}(x)\in uQ_1u^*\cap Q_2$ since uQ_1u^* and Q_2 form a commuting square. Altogether, $E_{Q_2}(x)\in Q_1\cap uQ_1u^*\cap Q_2$ for every $x\in Q_1\cap uQ_1u^*$. Hence $Q_1\cap uQ_1u^*$ and Q_2 form a commuting square inside M for every $u\in\mathcal{G}$.

Now let $y\in uQ_1u^*\cap Q_2$. Then $E_{Q_1\cap uQ_1u^*}(y)\in Q_1\cap uQ_1u^*\cap Q_2$, since $Q_1\cap uQ_1u^*$ and Q_2 form a commuting square inside M. So $Q_1\cap uQ_1u^*$ and $uQ_1u^*\cap Q_2$ also form a commuting square inside M for every $u\in\mathcal{G}$. In particular, $Q_1\cap uQ_1u^*$ and $uQ_1u^*\cap Q_2$ form a commuting square inside uQ_1u^* for every $u\in\mathcal{G}$.

Put $A := Q_1\cap uQ_1u^*$, $B := uQ_1u^*$ and $Q := Q_2$. Using Lemma 2.2.3, we get that $Q_1\cap uQ_1u^*\cap Q_2$ is a finite index subalgebra of $uQ_1u^*\cap Q_2$. Therefore

$$uQ_1u^*\cap Q_2 \prec^f_{uQ_1u^*\cap Q_2} Q_1\cap uQ_1u^*\cap Q_2.$$

By Lemma 1.10.10, this implies that $Q_1\cap uQ_1u^*\cap Q_2$ is a co-amenable subalgebra of $uQ_1u^*\cap Q_2$, thus ending the proof. $\square$

We used the following lemma in the proof of Corollary 2.3.2.

Lemma 2.3.3. *Let $(B,\tau) \subset \mathrm{B}(\mathcal{H})$ be a tracial von Neumann algebra. Let $A \subset B$ be a von Neumann subalgebra. Assume that $Q \subset \mathrm{B}(\mathcal{H})$ is a von Neumann algebra such that A and $B \cap Q$ form a commuting square in B. Then $[B \cap Q : A \cap Q] \leq [B : A]$.*

Proof. Since A and $B \cap Q$ form a commuting square in B, we have that $E_A(x) = E_{A\cap Q}(x)$ whenever $x \in B \cap Q$. So, the map $\psi(xe_{A\cap Q}y) = xe_A y$ for $x, y \in B \cap Q$ extends to a Tr-preserving, possibly nonunital, embedding $\psi : \langle B \cap Q, e_{A\cap Q}\rangle \to \langle B, e_A\rangle$. Therefore

$$[B \cap Q : A \cap Q] = \mathrm{Tr}_{\langle B\cap Q, e_{A\cap Q}\rangle}(1) = \mathrm{Tr}_{\langle B,e_A\rangle}(\psi(1)) \leq \mathrm{Tr}_{\langle B,e_A\rangle}(1) = [B : A].$$

□

We are now able to prove the following proposition.

Proposition 2.3.4. *Define $M = L(\Gamma)$ and $M_i = L(\hat{\Gamma}_i)$. Let n be a nonzero positive integer and let p be a nonzero projection of M^n. If $Q \subset pM^n p$ is a von Neumann subalgebra without an amenable direct summand, then there exists an $i \in \{1, \ldots, k\}$ such that $Q' \cap pM^n p \prec_{M^n} M_i$.*

Proof. Define $B_i = L(\Gamma_1 \times \ldots \times \Gamma_{i-1} \times \langle a_i^{n_i}\rangle \times \Gamma_{i+1} \times \ldots \times \Gamma_k)$. Let $z \in M^n$ be a nonzero projection and let $P \subset zM^n z$ be a von Neumann subalgebra. We have the following claim.

Claim. If $P \subset zM^n z$ is amenable relative to every B_i^n, then P is amenable.

Let us prove this by repeatedly using Corollary 2.3.2. To that end, let $\mathcal{G}_0$ be a countable subgroup of $\mathcal{U}_n(\mathbb{C})$ such that $\mathcal{G}_0'' = M_n(\mathbb{C})$. Put $\mathcal{G} = \{v \otimes u_g \mid v \in \mathcal{G}_0 \text{ and } g \in \Gamma\}$. Then $\mathcal{G} \leq \mathcal{U}(M^n)$ is a countable subgroup satisfying $\mathcal{G}'' = M^n$. Fix $u = v \otimes u_g \in \mathcal{G}$, with $v \in \mathcal{G}_0$ and $g = g_1 \times \ldots \times g_k \in \Gamma$. Then we have for every $s \in \{1, \ldots, k\}$ that

- $uB_s^n u^* = L(\Gamma_1 \times \ldots \times \Gamma_{s-1} \times g_s\langle a_s^{n_s}\rangle g_s^{-1} \times \Gamma_{s+1} \times \ldots \times \Gamma_k)^n$;
- $B_s^n \cap uB_s^n u^* = L(\Gamma_1 \times \ldots \times \Gamma_{s-1} \times (\langle a_s^{n_s}\rangle \cap g_s\langle a_s^{n_s}\rangle g_s^{-1}) \times \Gamma_{s+1} \times \ldots \times \Gamma_k)^n$.

So $B_s^n \cap uB_s^n u^*$ is a finite index subalgebra of B_s^n. Now let $J \subset \{1, \ldots, k\}$ be a subset and define

$$\Gamma_i^{(J)} = \begin{cases} \langle a_i^{n_i}\rangle & \text{if } i \in J, \\ \Gamma_i & \text{if } i \notin J. \end{cases}$$

Then $\bigcap_{t\in J} B_t^n = (\bigcap_{t\in J} B_t)^n = L(\Gamma_1^{(J)} \times \ldots \times \Gamma_k^{(J)})^n$. So $uB_s^n u^*$ and $\bigcap_{t\in J} B_t^n$ form a commuting square inside M^n, as is described before Proposition 2.3.1. Altogether, we see that the subalgebras B_s^n and $\bigcap_{t\in J} B_t^n$ of M^n together with $\mathcal{G}$ satisfy all three conditions in Corollary 2.3.2. An induction argument now shows that if $P \subset zM^n z$ is amenable relative to every B_i^n inside M^n, then P is amenable relative to $\bigcap_i B_i^n$ inside M^n. Now $\bigcap_i B_i^n = L(\langle a^{n_1}\rangle \times \ldots \times \langle a^{n_k}\rangle)^n$ is amenable. From the second statement of Lemma 1.9.6, we therefore find that Qz is amenable. This ends the proof of the claim.

Let us now prove that for every nonzero projection $z \in \mathcal{Z}(Q' \cap pM^n p)$ there exists an $i \in \{1, \ldots, k\}$ such that $Qz \subset zM^n z$ is not amenable relative to B_i^n. So let $z \in \mathcal{Z}(Q' \cap pM^n p)$ be a nonzero projection and assume, by way of reaching a contradiction, that $Qz \subset zM^n z$ is amenable relative to every B_i^n. By our claim, we have that Qz is amenable. Since $Qz \cong Q\,\mathrm{supp}(E_Q(z)) = Q\,\mathrm{supp}(E_{\mathcal{Z}(Q)}(z))$, this contradicts the fact that Q has no amenable direct summand. We conclude that indeed for every nonzero projection $z \in \mathcal{Z}(Q' \cap pM^n p)$ there exist an $i \in \{1, \ldots, k\}$ such that $Qz \subset zM^n z$ is not amenable relative to B_i^n.

Now define for every nonzero projection $z \in \mathcal{Z}(Q' \cap pM^n p)$ the set

$$\mathcal{F}(z) := \{i \in \{1, \ldots, k\} \mid Qz \subset zM^n z \text{ is not amenable relative to } B_i^n\}.$$

We already showed that $\mathcal{F}(z) \neq \emptyset$ for every z. Furthermore we have that $\mathcal{F}(z') \subset \mathcal{F}(z)$ whenever $z' \leq z$. Now fix a nonzero projection $q \in \mathcal{Z}(Q' \cap pM^n p)$ such that $\mathcal{F}(q)$ is minimal. Then we can fix an $i \in \{1, \ldots, k\}$ such that $Qz \subset zM^n z$ is not amenable relative to B_i^n for every nonzero projection $z \in \mathcal{Z}((Qq)' \cap qM^n q)$.

Let us finish the proof of the proposition. By Lemma 1.12.3, we have that $M^n = \mathrm{HNN}(M_i^n, B_i^n, \theta)$. Apply Proposition 1.12.5 to $Qq \subset qM^n q$. Since we already excluded the third possibility, we have that

$$(Qq)' \cap qM^n q \prec_{M^n} B_i^n \text{ or } N_{qM^n q}(Qq)'' \prec_{M^n} M_i^n.$$

Now since $B_i^n \subset M_i^n$ and $(Qq)' \cap qM^n q \subset N_{qM^n q}(Qq)''$, both intertwinings imply that $(Qq)' \cap qM^n q \prec_{M^n} M_i^n$. By Lemma 1.10.3 we can conclude that $Q' \cap pM^n p \prec_{M^n} M_i$. $\square$

Theorem 2.3.7 below concludes the first part of the proof of Theorem A. Before we go into that, we recall some notation from [Va08] and prove a small lemma.

Notation 2.3.5 ([Va08, Notation 3.10]). Let (M, τ_M) and (N, τ_N) be tracial von Neumann algebras and $A \subset M$, $B \subset N$ von Neumann subalgebras. Let ${}_M\mathcal{H}_N$ be an M-N-bimodule.

- We set $A \prec_{\mathcal{H}} B$ if $\mathcal{H}$ contains a non-zero A-B-subbimodule $\mathcal{K}$ with $\dim_{-B}(\mathcal{K}) < \infty$.
- We set $A \prec^f_{\mathcal{H}} B$ if every nonzero A-N-subbimodule $\mathcal{K} \subset \mathcal{H}$ satisfies $A \prec_{\mathcal{K}} B$.

Lemma 2.3.6. *Let (M, τ_M) and (N, τ_N) be tracial von Neumann algebras and $A \subset M$, $B \subset N$ von Neumann subalgebras. Let ${}_M\mathcal{H}_N$ be an M-N-bimodule. If ${}_M\mathcal{H}_N \cong {}_{\psi(M)}p(\mathbb{C}^n \otimes L^2(N))_N$ for some $*$-homomorphism $\psi : M \to pN^np$ and a projection $p \in N^n$, then*

- *$A \prec_{\mathcal{H}} B$ if and only if $\psi(A) \prec_{N^n} B$,*
- *$A \prec^f_{\mathcal{H}} B$ if and only if $\psi(A) \prec^f_{N^n} B$.*

Proof. Assume that ${}_M\mathcal{H}_N \cong {}_{\psi(M)}p(\mathbb{C}^n \otimes L^2(N))_N$ for some $*$-homomorphism $\psi : M \to pN^np$ and a projection $p \in N^n$.

To prove the first statement we first observe that

$$ {}_{\psi(A)}pL^2(N^n)_B \cong \bigoplus_{i=1}^{n} {}_{\psi(A)}p(\mathbb{C}^n \otimes L^2(N))_B \cong \bigoplus_{i=1}^{n} {}_A\mathcal{H}_B. \tag{2.7} $$

Now assume that $A \prec_{\mathcal{H}} B$. Then there exists a nonzero A-B-subbimodule $\mathcal{K}$ of $\mathcal{H}$ with finite B-dimension. By viewing $\mathcal{H}$ as an A-B-subbimodule of $\bigoplus_{i=1}^n \mathcal{H}$, we see that from (2.7) that there exists a nonzero $\psi(A)$-B-subbimodule of $pL^2(N^n)$ with finite B-dimension. So $\psi(A) \prec_{N^n} B$, showing one implication.

Now assume that $\psi(A) \prec_{N^n} B$. Then there exists a nonzero $\psi(A)$-B-subbimodule $\mathcal{K}$ of $pL^2(N^n)$ with finite B-dimension. By (2.7) we can view $\mathcal{K}$ as an A-B-subbimodule of $\bigoplus_{i=1}^n \mathcal{H}$. If we then project it onto an appropriate copy of $\mathcal{H}$ we find a nonzero A-B-subbimodule of $\mathcal{H}$ with finite B-dimension. So $A \prec_{\mathcal{H}} B$, showing also the converse implication.

To prove the second statement we first make the following observation. Every A-N-subbimodule $\mathcal{K}$ of $\mathcal{H}$ is isomorphic with

$$ {}_{\psi_q(A)}q(\mathbb{C}^n \otimes L^2(N))_B, $$

where $q \in \psi(A)' \cap pN^np$ is a projection and $\psi_q(\cdot) = \psi(\cdot)q$. Combining this observation with the first statement we get that $A \prec_{\mathcal{K}} B$ for every A-N-subbimodule $\mathcal{K} \subset \mathcal{H}$ if and only if $\psi(A)q \prec_{N^n} B$ for every projection $q \in \psi(A)' \cap pN^np$. This then immediately implies that $A \prec^f_{\mathcal{H}} B$ if and only if $\psi(A) \prec^f_{N^n} B$. □

Theorem 2.3.7. *Let $M = L(\Gamma)$, $M_i = L(\hat{\Gamma}_i)$, $N = L(\Lambda)$ and $N_j = L(\hat{\Lambda}_j)$. Assume that there exists a nonzero finite index M-N-bimodule $\mathcal{H}$. Then $k = l$ and there exists a permutation $\sigma \in \mathrm{Sym}(k)$ such that for every $i \in \{1, \ldots, k\}$ there exists a nonzero finite index $\mathcal{Z}(M_i)$-$\mathcal{Z}(N_{\sigma(i)})$-subbimodule of $\mathcal{H}$.*

Proof. By interchanging if necessary the roles of M and N, we may assume that $k \geq l$. Now let ${}_M\mathcal{H}_N$ be a nonzero finite index M-N-bimodule. Since M and N are factors, we may assume that $\mathcal{H}$ is irreducible, i.e. $\mathcal{H}$ has no nontrivial M-N-subbimodules. As usual there exists

- a nonzero positive integer n;
- a nonzero projection $p \in N^n$;
- a normal $*$-homomorphism $\psi : M \to pN^n p$,

such that $\psi(M) \subset pN^n p$ is irreducible and ${}_M\mathcal{H}_N \cong {}_{\psi(M)}p(\mathbb{C}^n \otimes L^2(N))_N$. For every $i \in \{1, \ldots, k\}$ denote by $\mathcal{C}_i \leq \Gamma_i$ the centralizer of $\langle a_i^{n_i} \rangle$ inside Γ_i. By Lemma 1.11.2 and Lemma 1.9.4, we have that $L(\mathcal{C}_i)$ has no amenable direct summand. Applying Proposition 2.3.4 to the inclusion $\psi(L(\mathcal{C}_i)) \subset pN^n p$, we find that there exists some $j \in \{1, \ldots, l\}$ such that $\psi(L(\mathcal{C}_i))' \cap pN^n p \prec_{N^n} N_j$. Now write $B_i = L(\Gamma_1 \times \ldots \times \Gamma_{i-1} \times \langle a_i^{n_i} \rangle \times \Gamma_{i+1} \times \ldots \times \Gamma_k)$. Since $\psi(B_i)$ is a von Neumann subalgebra of $\psi(L(\mathcal{C}_i))' \cap pN^n p$ we have that $\psi(B_i) \prec_{N^n} N_j$. And since $B_i \subset M_i$ is a finite index inclusion, Lemma 1.10.3 implies that there exists some $j \in \{1, \ldots, l\}$ such that $\psi(M_i) \prec_{N^n} N_j$. Altogether, for every $i \in \{1, \ldots, k\}$ there exists some $j \in \{1, \ldots, l\}$ such that $\psi(M_i) \prec_{N^n} N_j$.

On the other hand, there also exists

- a nonzero positive integer m;
- a nonzero projection $q \in M^m$;
- a normal $*$-homomorphism $\varphi : N \to qM^m q$,

such that $\varphi(N) \subset qM^m q$ is irreducible and ${}_M\mathcal{H}_N \cong {}_M(L^2(M) \otimes \mathbb{C}^m)q_{\varphi(N)}$. By symmetry, also for every $j \in \{1, \ldots, l\}$ there exists some $i \in \{1, \ldots, k\}$ such that $\varphi(N_j) \prec_{M^m} M_i$.

Claim. If $\psi(M_{i_1}) \prec_{N^n} N_j$ and $\varphi(N_j) \prec_{M^m} M_{i_2}$, then $i_1 = i_2$.

So assume that $\psi(M_{i_1}) \prec_{N^n} N_j$ and $\varphi(N_j) \prec_{M^m} M_{i_2}$. We have that $\hat{\Gamma}_{i_1} \leq \Gamma$ and $\hat{\Lambda}_j \leq \Lambda$ are almost normal subgroups. So by Lemma 1.11.4, $M_{i_1} \subset M$ and

$N_j \subset N$ are quasi-regular. Since also $\psi(M) \subset pN^n p$ and $\varphi(N) \subset qM^m q$ are irreducible, Lemma 1.10.8 implies that $\psi(M_{i_1}) \prec^f_{N^n} N_j$ and $\varphi(N_j) \prec^f_{M^m} M_{i_2}$.

Since $\psi(M_{i_1}) \prec_{N^n} N_j$, we find that $(\mathrm{id} \otimes \varphi)(\psi(M_{i_1})) \prec_{N^{mn}} \varphi(N_j)$. We already found that $\varphi(N_j) \prec^f_{M^m} M_{i_2}$. Combining the two intertwinings using Lemma 1.10.6 yields

$$(\mathrm{id} \otimes \varphi)(\psi(M_{i_1})) \prec^f_{M^{mn}} M_{i_2}. \tag{2.8}$$

On the other hand, we have that

$$_M(\mathcal{H} \otimes_N \overline{\mathcal{H}})_M \cong {}_{(\mathrm{id} \otimes \varphi)(\psi(M))} p_0(\mathbb{C}^{mn} \otimes L^2(M))_M, \tag{2.9}$$

where $p_0 := (\mathrm{id} \otimes \varphi)(p)$. Combining (2.8), (2.9) and Lemma 2.3.6, we get that

$$M_{i_1} \prec^f_{(\mathcal{H} \otimes_N \overline{\mathcal{H}})} M_{i_2}. \tag{2.10}$$

Let us now show that $\mathcal{H} \otimes_N \overline{\mathcal{H}}$ contains an M-M-subbimodule isomorphic with $L^2(M)$. To that end, note that $_M(\mathcal{H} \otimes_N \overline{\mathcal{H}})_M \cong {}_{\psi(M)}L^2(pN^n p)_{\psi(M)}$. Since M is a factor we also have that $_M L^2(M)_M \cong {}_{\psi(M)}L^2(\psi(M))_{\psi(M)}$. Finally we have that $L^2(\psi(M))$ is a $\psi(M)$-$\psi(M)$-subbimodule of $L^2(pN^n p)$. Altogether we see that $\mathcal{H} \otimes_N \overline{\mathcal{H}}$ indeed contains an M-M-subbimodule isomorphic with $L^2(M)$. Combining this with (2.10), we get that $L^2(M)$ must contain a nonzero M_{i_1}-M_{i_2}-subbimodule that is finitely generated as an M_{i_2}-module. In other words, we have found that $M_{i_1} \prec_M M_{i_2}$.

We now show that $M_{i_1} \prec_M M_{i_2}$ implies that $L(\Gamma_{i_2}) \prec_{L(\Gamma_{i_2})} L(\langle a_{i_2} \rangle)$, whenever $i_1 \neq i_2$. To that end, assume by way of reaching a contradiction that $L(\Gamma_{i_2}) \not\prec_{L(\Gamma_{i_2})} L(\langle a_{i_2} \rangle)$ and $i_1 \neq i_2$. By Theorem 1.10.1, there exist a sequence of unitaries $u_n \in \mathcal{U}(L(\Gamma_{i_2}))$ satisfying $||E_{L(\langle a_{i_2} \rangle)}(x u_n y^*)||_2 \to 0$ for all $x, y \in L(\Gamma_{i_2})$. Since $i_1 \neq i_2$, we have that $L(\Gamma_{i_2})$ is a von Neumann subalgebra of M_{i_1}. So $(u_n)_n$ can be viewed as a sequence of unitaries in $\mathcal{U}(M_{i_1})$. Moreover $||E_{M_{i_2}}(x u_n y^*)||_2 \to 0$ for all $x, y \in M$. Using Theorem 1.10.1 again, this means that $M_{i_1} \not\prec_M M_{i_2}$. This is a contradiction and so $L(\Gamma_{i_2}) \prec_{L(\Gamma_{i_2})} L(\langle a_{i_2} \rangle)$ if $i_1 \neq i_2$.

Finally note that $L(\Gamma_{i_2})$ is a nonamenable factor and that $L(\langle a_{i_2} \rangle)$ is amenable. This implies that $L(\Gamma_{i_2}) \prec_{L(\Gamma_{i_2})} L(\langle a_{i_2} \rangle)$ cannot hold. Therefore $i_1 = i_2$, ending the proof of the claim.

Let us continue with the proof of the theorem. Using the claim together with Lemma 2.3.6, we have so far found that for every $i \in \{1, \dots, k\}$ there exists some $j \in \{1, \dots, l\}$ such that $\mathcal{H}$ contains a nonzero M_i-N_j-subbimodule that is finitely generated as an N_j-module and also contains a nonzero M_i-N_j-subbimodule that is finitely generated as an M_i-module. Similar to Proposition 1.10.9, the quasi-regularity of M_i and N_j now implies that for every $i \in \{1, \dots, k\}$ there exists some $j \in \{1, \dots, l\}$ such that $\mathcal{H}$ contains a nonzero bifinite M_i-N_j-subbimodule.

To end the proof we choose a map $\sigma : \{1, \ldots, k\} \to \{1, \ldots, l\}$ in such a way that $\mathcal{H}$ contains a nonzero bifinite M_i-$N_{\sigma(i)}$-subbimodule for every $i \in \{1, \ldots, k\}$. Using the claim and Lemma 2.3.6 again, we see that σ must be injective. Since we assumed that $k \geq l$, we actually have that $k = l$ and that $\sigma \in \mathrm{Sym}(k)$. Lemma 2.3.8 now finishes the proof. $\square$

We made use of the following lemma in the proof of Theorem 2.3.7.

Lemma 2.3.8. *Let (M, τ_M) and (N, τ_N) be tracial von Neumann algebras and let ${}_M\mathcal{H}_N$ be a nonzero bifinite M-N-bimodule. Then there exists a nonzero bifinite $\mathcal{Z}(M)$-$\mathcal{Z}(N)$-subbimodule ${}_{\mathcal{Z}(M)}\mathcal{K}_{\mathcal{Z}(N)}$ of ${}_{\mathcal{Z}(M)}\mathcal{H}_{\mathcal{Z}(N)}$.*

Proof. Let (M, τ_M) and (N, τ_N) be tracial von Neumann algebras and let ${}_M\mathcal{H}_N$ be a nonzero bifinite M-N-bimodule. Then there exists

- a projection $p \in N^\infty$ with $\mathrm{Tr}(p) < \infty$
- a normal $*$-homomorphism $\psi : M \to pN^\infty p$ with $[pN^\infty p : \psi(M)] < \infty$

such that ${}_M\mathcal{H}_N \cong {}_{\psi(M)}p(l^2(\mathbb{N}) \otimes L^2(N))_N$.

Using [Va08, Lemma A.3], we get that $\psi(\mathcal{Z}(M)) \subset \psi(M)' \cap pN^\infty p$ is also a finite index inclusion. Therefore we can take a nonzero projection $q \in \psi(M)' \cap pN^\infty p$ such that $q(\psi(M)' \cap pN^\infty p)q = \psi(\mathcal{Z}(M))q$. This in turn gives us a nonzero M-N-subbimodule $\mathcal{H}_1$ of $\mathcal{H}$ where $\mathrm{B}({}_M\mathcal{H}_{1N})$ coincides with the left $\mathcal{Z}(M)$-action.

Analogously, but replacing $\mathcal{H}$ with $\mathcal{H}_1$, we can find a nonzero M-N-subbimodule $\mathcal{H}_2$ of $\mathcal{H}_1$ where $\mathrm{B}({}_M\mathcal{H}_{2N})$ coincides with the right $\mathcal{Z}(N)$-action. Since ${}_M\mathcal{H}_{2N} \subset {}_M\mathcal{H}_{1N}$, we also still have that $\mathrm{B}({}_M\mathcal{H}_{2N})$ coincides with the left $\mathcal{Z}(M)$-action.

Now take $\xi \in \mathcal{H}_2 \setminus \{0\}$. We have just showed that $\mathcal{Z}(M)\xi = \xi\mathcal{Z}(N)$. Hence ${}_{\mathcal{Z}(M)}\overline{\mathcal{Z}(M)\xi}_{\mathcal{Z}(N)}$ is a nonzero bifinite $\mathcal{Z}(M)$-$\mathcal{Z}(N)$-subbimodule. This ends the proof. $\square$

2.3.2 Second step of the proof

We start with the following observation.

Proposition 2.3.9. *Let $n, m \in \mathbb{Z}$ satisfy $2 \leq n < |m|$. Define $M := L(\mathrm{BS}(n, m))$ and $A := L(\langle a \rangle)$. We have that $A \subset M$ is a quasi-regular abelian von Neumann subalgebra satisfying $\mathcal{Z}(A' \cap M) = A$.*

Proof. Since $\langle a\rangle$ is an almost normal subgroup of $\mathrm{BS}(n,m)$, we get from Lemma 1.11.4 that A is a quasi-regular von Neumann subalgebra of M. On the other hand, since $\langle a\rangle$ is an abelian group, we have that A is an abelian von Neumann algebra.

To prove that $\mathcal{Z}(A'\cap M) = A$, we define the finite index subalgebra $A_0 := L(\langle a^n\rangle)$ of A. We first prove that $\mathcal{Z}(A_0'\cap M) = A_0$. Afterwards we will show that this implies that $\mathcal{Z}(A'\cap M) = A$.

Define $G := \langle a^{\mathbb{Z}}, b^{-1}a^{\mathbb{Z}}b\rangle \subset \mathrm{BS}(n,m)$. Then $L(G)$ is a subalgebra of $A_0'\cap M$. So $\mathcal{Z}(A_0'\cap M) \subset L(G)'\cap M$. Using Lemma 1.11.1, one can easily see that $\{g\gamma g^{-1} \mid g \in G\}$ is an infinite set for every $\gamma \in \mathrm{BS}(n,m) \setminus \langle a^n\rangle$. From this we now show that $L(G)'\cap M \subset A_0$ by using Murray and von Neumann's conjugacy class technique (see [MvN43, Lemma 5.3.4]).

Let $x \in L(G)'\cap M$. Write $x = \sum_{\gamma\in\mathrm{BS}(n,m)} c_\gamma u_\gamma$, where the sum converges in $||\cdot||_2$-norm. By assumption, we have that $x = u_g x u_g^*$ for every $g \in G$. Therefore $x = \sum c_\gamma u_{g\gamma g^{-1}}$ for every $g \in G$. By comparing the two decompositions, we find that $c_\gamma = c_{g\gamma g^{-1}}$ for every $\gamma \in \mathrm{BS}(n,m)$ and $g \in G$. If $\{g\gamma g^{-1} \mid g \in G\}$ is an infinite set for some $\gamma \in \mathrm{BS}(n,m)$, we have that c_γ is zero, since otherwise the sum $\sum c_\gamma u_\gamma$ cannot converge in $||\cdot||_2$-norm. From this we get that c_γ can only be nonzero when γ is an element of $\langle a^n\rangle$. In other words, we indeed have that $L(G)'\cap M \subset A_0$.

Since we already showed that $\mathcal{Z}(A_0'\cap M) \subset L(G)'\cap M$, we have proved that $\mathcal{Z}(A_0'\cap M) \subset A_0$. Since the converse inclusion is obvious, we find that $A_0 = \mathcal{Z}(A_0'\cap M)$.

Since $A_0 \subset A$ has finite index, there exist orthogonal projections $p_j \in A$ such that $Ap_j = A_0 p_j$ and $\sum_j p_j = 1$. But then

$$\begin{aligned}\mathcal{Z}(A'\cap M)p_j &= \mathcal{Z}((A'\cap M)p_j) = \mathcal{Z}((Ap_j)'\cap p_j M p_j)\\ &= \mathcal{Z}((A_0 p_j)'\cap p_j M p_j) = \mathcal{Z}(p_j(A_0'\cap M)p_j)\\ &= \mathcal{Z}(A_0'\cap M)p_j = A_0 p_j \subset A\end{aligned}$$

Therefore $\mathcal{Z}(A'\cap M) \subset A$. The converse inclusion being obvious, we have proven that $A = \mathcal{Z}(A'\cap M)$. □

We now identify the countable equivalence relation $\mathcal{R}(A \subset M)$, when $M = L(\mathrm{BS}(n,m))$ and $A = L(\langle a\rangle)$.

Proposition 2.3.10. *Let $n,m \in \mathbb{Z}$ satisfy $2 \leq n < |m|$. Define $M = L(\mathrm{BS}(n,m))$ and $A = L(\langle a\rangle)$. The equivalence relation $\mathcal{R}(A \subset M)$ is isomorphic with the unique hyperfinite ergodic countable equivalence relation of type $\mathrm{III}_{n/|m|}$.*

Proof. Let $\mathcal{R}_0 = \{(y,z) \in \mathbb{T} \times \mathbb{T} \mid y^m = z^n\}$. Denote by $\mathcal{R}_{n,m}$ the equivalence relation on $\mathbb{T}$ generated by $\mathcal{R}_0$. Recall from Example 2.2.5 that $\mathcal{R}_{n,m}$ is the unique hyperfinite ergodic countable equivalence relation of type $\mathrm{III}_{n/|m|}$. Define $\pi : \mathcal{R}_0 \to \mathbb{T} : \pi(y,z) = y^m$. Note that π is $n|m|$-to-1. Define the probability measure μ on $\mathcal{R}_0$ given by

$$\mu(U) = \frac{1}{n|m|} \int_{\mathbb{T}} \#\big(U \cap \pi^{-1}(\{x\})\big) \, d\lambda(x) \, .$$

For all $k, l \in \mathbb{Z}$, we define the function $P_{k,l} : \mathcal{R}_0 \to \mathbb{T} : P_{k,l}(y,z) = y^k z^l$. We have the following claim.

Claim. $\langle P_{k,l}, P_{s,t} \rangle = \langle u_a^k u_b u_a^l, u_a^s u_b u_a^t \rangle$ **for all** $k, l, s, t \in \mathbb{Z}$**.**

We begin with the following calculation:

$$\begin{aligned}
\langle P_{k,l}, P_{s,t} \rangle &= \int_{\mathcal{R}_0} y^{k-s} z^{l-t} \, d\mu(y,z) \\
&= \frac{1}{n|m|} \int_{\mathbb{T}} \left(\sum_{(y,z) \in \pi^{-1}(x)} y^{k-s} z^{l-t} \right) d\lambda(x) \\
&= \frac{1}{n|m|} \int_0^1 \left(\sum_{p=1}^{|m|} \sum_{q=1}^{n} e^{(2\pi i x \frac{(k-s)}{m} + 2\pi i p \frac{(k-s)}{m})} e^{(2\pi i x \frac{(l-t)}{n} + 2\pi i q \frac{(l-t)}{n})} \right) dx \\
&= \frac{1}{n|m|} \sum_{p=1}^{|m|} e^{2\pi i p \frac{(k-s)}{m}} \sum_{q=1}^{n} e^{2\pi i q \frac{(l-t)}{n}} \int_0^1 e^{2\pi i x (\frac{(k-s)}{m} + \frac{(l-t)}{n})} \, dx.
\end{aligned}$$

Furthermore we have that

$$\sum_{p=1}^{|m|} e^{2\pi i p \frac{(k-s)}{m}} = \begin{cases} |m| & \text{if } m \mid (k-s) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sum_{q=1}^{n} e^{2\pi i q \frac{(l-t)}{n}} = \begin{cases} n & \text{if } n \mid (l-t) \\ 0 & \text{otherwise.} \end{cases}$$

Also, if both $m \mid (k-s)$ and $n \mid (l-t)$, then

$$\int_0^1 e^{2\pi i x (\frac{(k-s)}{m} + \frac{(l-t)}{n})} \, dx = \begin{cases} 1 & \text{if } \frac{(k-s)}{m} + \frac{(l-t)}{n} = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Altogether we get that

$$\langle P_{k,l}, P_{s,t}\rangle = \begin{cases} 1 & \text{if } n \mid (l-t) \text{ and } \frac{(k-s)}{m} + \frac{(l-t)}{n} = 0 \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, we have that

$$\langle u_a^k u_b u_a^l, u_a^s u_b u_a^t\rangle = \tau(u_a^k u_b u_a^{l-t} u_b^{-1} u_a^{-s}).$$

By Lemma 1.11.1, $\tau(u_a^k u_b u_a^{l-t} u_b^{-1} u_a^{-s}) = 0$ whenever $n \nmid (l-t)$. Now if $n \mid (l-t)$, then

$$\begin{aligned} \langle u_a^k u_b u_a^l, u_a^s u_b u_a^t\rangle &= \tau(u_a^{(k-s)+m(l-t)/n}) \\ &= \begin{cases} 1 & \text{if } \frac{(k-s)}{m} + \frac{(l-t)}{n} = 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Altogether, we also get that

$$\langle u_a^k u_b u_a^l, u_a^s u_b u_a^t\rangle = \begin{cases} 1 & \text{if } n \mid (l-t) \text{ and } \frac{(k-s)}{m} + \frac{(l-t)}{n} = 0 \\ 0 & \text{otherwise.} \end{cases}$$

This proves the claim.

Let us continue with the proof of the proposition. Using the claim, we can define a unitary T by

$$T : L^2(\mathcal{R}_0, \mu) \to \overline{\text{span}}^{\|\cdot\|_2} A u_b A : P_{k,l} \mapsto u_a^k u_b u_a^l \; .$$

We turn $L^2(\mathcal{R}_0, \mu)$ into an $L^\infty(\mathbb{T})$-$L^\infty(\mathbb{T})$-bimodule by the formula

$$(F \cdot \xi \cdot F')(y,z) = F(y)\, \xi(y,z)\, F'(z) \; .$$

Under the natural identification of $L^\infty(\mathbb{T})$ and A, the unitary T is A-A-bimodular.

By construction ${}_A L^2(\mathcal{R}_0, \mu)_A$ is isomorphic with a direct sum of bimodules of the form ${}_A\mathcal{H}(\alpha_j)_A$ where the union of the graphs of the partial automorphisms α_j equals $\mathcal{R}_0$ and hence generates the equivalence relation $\mathcal{R}_{n,m}$. Applying Lemma 2.1.8 to $\mathcal{F} = \{u_b\}$, we conclude that $\mathcal{R}(A \subset M) \cong \mathcal{R}_{n,m}$ up to measure zero. □

Finally we can give the actual proof of the main theorem.

Proof of Theorem A. Let k and l be nonzero positive integers. For every $i \in \{1, \ldots, k\}$ and every $j \in \{1, \ldots, l\}$, let $n_i, m_i, p_j, q_j \in \mathbb{Z}$ satisfy $2 \leq n_i < |m_i|$ and $2 \leq p_j < |q_j|$. Define $M := L(\Gamma)$ and $N := L(\Lambda)$ and assume that ${}_M\mathcal{H}_N$ is a finite index M-N-bimodule. Write $x := \dim_{M-}(\mathcal{H})$ and $y := \dim_{-N}(\mathcal{H})$. We need to prove that $k = l$ and that there exists a permutation $\sigma \in \mathrm{Sym}(k)$ such that for every $i \in \{1, \ldots, k\}$, we have

$$(n_i/|m_i|)^{r_i} = (p_{\sigma(i)}/|q_{\sigma(i)}|)^{s_i} \text{ for some } r_i, s_i \in \mathbb{Z} \text{ with } 1 \leq r_i, s_i \leq xy. \quad (2.11)$$

First of all, write $M_i := L(\Gamma_i)$ and $N_j := L(\Lambda_j)$. By Theorem 2.3.7, we have that $k = l$ and that there exists a permutation $\sigma \in \mathrm{Sym}(k)$ such that for every $i \in \{1, \ldots, k\}$ there exists a nonzero finite index $\mathcal{Z}(M_i)$-$\mathcal{Z}(N_{\sigma(i)})$-subbimodule of $\mathcal{H}$. Theorem 2.2.6 then implies that there exists a finite index correspondence between $\mathcal{R}(\mathcal{Z}(M_i) \subset M)$ and $\mathcal{R}(\mathcal{Z}(N_{\sigma(i)}) \subset N)$ with left and right index smaller than xy.

Let us now show that

$$\mathcal{R}(\mathcal{Z}(M_i) \subset M) \cong \mathcal{R}(L(\langle a_i \rangle) \subset L(BS(n_i, m_i)))$$

and

$$\mathcal{R}(\mathcal{Z}(N_{\sigma(i)}) \subset N) \cong \mathcal{R}(L(\langle c_{\sigma(i)} \rangle) \subset L(BS(p_{\sigma(i)}, q_{\sigma(i)}))).$$

To that end, let $\mathcal{J} \subset Q_{L(BS(n_i,m_i))}(L(\langle a_i \rangle))$ be a countable $||\cdot||_2$-dense subset. Recall that

$$\mathcal{R}(L(\langle a_i \rangle) \subset L(BS(n_i, m_i))) = \bigcup_{v \in \mathcal{J}} \mathrm{graph}(\varphi_{\alpha_v}).$$

Set $\tilde{\mathcal{J}} = \{u_{g_1} \otimes \ldots \otimes u_{g_{i-1}} \otimes v \otimes u_{g_{i+1}} \otimes \ldots \otimes u_{g_k} \mid v \in \mathcal{J} \text{ and } g_j \in \mathrm{BS}(n_j, m_j)\}$. Then $\tilde{\mathcal{J}}$ is countable and $||\cdot||_2$-norm densely spans $L^2(M)$. Therefore

$$\mathcal{R}(\mathcal{Z}(M_i) \subset M) = \bigcup_{w \in \tilde{\mathcal{J}}} \mathrm{graph}(\varphi_{\alpha_w}).$$

Let $w = u_{g_1} \otimes \ldots \otimes u_{g_{i-1}} \otimes v \otimes u_{g_{i+1}} \otimes \ldots \otimes u_{g_k}$ be an element of $\tilde{\mathcal{J}}$. Under the natural identification of $\mathcal{Z}(M_i)$ and $L(\langle a_i \rangle)$, we have that $\alpha_w = \alpha_v$. Altogether, we see that

$$\mathcal{R}(\mathcal{Z}(M_i) \subset M) \cong \mathcal{R}(L(\langle a_i \rangle) \subset L(BS(n_i, m_i))).$$

Completely analogous, we have that

$$\mathcal{R}(\mathcal{Z}(N_{\sigma(i)}) \subset N) \cong \mathcal{R}(L(\langle c_{\sigma(i)} \rangle) \subset L(BS(p_{\sigma(i)}, q_{\sigma(i)}))).$$

Let us now finish the proof of the theorem. So far we have found that there exists a finite index correspondence between $\mathcal{R}(L(\langle a_i \rangle) \subset L(BS(n_i, m_i)))$ and $\mathcal{R}(L(\langle c_{\sigma(i)} \rangle) \subset L(BS(p_{\sigma(i)}, q_{\sigma(i)})))$ with left and right index smaller than xy. Using Proposition 2.2.4 and Proposition 2.3.10 we find (2.11). $\square$

Chapter 3

A rigidity result for crossed products of actions of Baumslag-Solitar groups

This chapter will be dedicated to proving our second main result which is Theorem C below. Let us first give some more background involving the setting.

Measure equivalence was introduced by Gromov in [Gr93]. Two countable discrete groups Γ and Λ are called *measure equivalent* if there exist ergodic essentially free pmp actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$ that are stably orbit equivalent. By the work of Ornstein and Weiss in [OW80], we know that all infinite amenable groups are measure equivalent to each other. On the other hand, it is well known (c.f. [Zi84, Proposition 4.3.3]) that nonamenable groups are not measure equivalent to amenable ones. Therefore the measure equivalence class of $\mathbb{Z}$ is exactly the class of all infinite amenable groups. In the nonamenable case, it is in general very hard to determine whether two nonisomorphic groups are measure equivalent.

Although the question asking whether two nonisomorphic nonamenable Baumslag-Solitar groups are measure equivalent is still open, Kida obtained a measure equivalence rigidity result for nonamenable Baumslag-Solitar groups in [Ki11]. Recently, that result was generalized by Houdayer and Raum in [HR13]. It goes as follows. Let $n_1, m_1, n_2, m_2 \in \mathbb{Z}$ such that $2 \leq n_1 \leq |m_1|$ and $2 \leq n_2 \leq |m_2|$. They proved that if $\mathrm{BS}(n_1, m_1)$ and $\mathrm{BS}(n_2, m_2)$ have stably orbit equivalent ergodic essentially free pmp actions such that the canonical abelian almost normal subgroups $\langle a_1 \rangle$ and $\langle a_2 \rangle$ act aperiodically (i.e. every finite index subgroup acts ergodically), then

- $n_1 = n_2$ and $m_1 = m_2$, if $n_1 \neq |m_1|$;
- $n_1 = n_2$ and $|m_1| = |m_2|$, if $n_1 = |m_1|$.

This brings us to our second main result.

Theorem C. *Let $n_1, m_1, n_2, m_2 \in \mathbb{Z}$ such that $2 \leq n_1 \leq |m_1|$ and $2 \leq n_2 \leq |m_2|$. For $i \in \{1,2\}$, let (P_i, τ_i) be a diffuse amenable tracial von Neumann algebra and let $\mathrm{BS}(n_i, m_i) \curvearrowright P_i$ be a trace preserving action such that the subalgebra $P_i \rtimes \langle a_i^{l(g)} \rangle \subset P_i \rtimes \mathrm{BS}(n_i, m_i)$ is irreducible for every $g \in \mathrm{BS}(n_i, m_i)$. If the crossed products $P_1 \rtimes \mathrm{BS}(n_1, m_1)$ and $P_2 \rtimes \mathrm{BS}(n_2, m_2)$ are stably isomorphic, then*

- *$n_1 = n_2$ and $m_1 = m_2$, if $n_1 \neq |m_1|$;*
- *$n_1 = n_2$ and $|m_1| = |m_2|$, if $n_1 = |m_1|$.*

We have the following corollary that generalizes the result of Houdayer and Raum mentioned above.

Corollary D. *Let $n_1, m_1, n_2, m_2 \in \mathbb{Z}$ such that $2 \leq n_1 \leq |m_1|$ and $2 \leq n_2 \leq |m_2|$. For $i \in \{1,2\}$, let $\mathrm{BS}(n_i, m_i) \curvearrowright (X_i, \mu_i)$ be an ergodic essentially free pmp action such that $\langle a_i \rangle \curvearrowright X_i$ is aperiodic. If the crossed products $L^\infty(X_1) \rtimes \mathrm{BS}(n_1, m_1)$ and $L^\infty(X_2) \rtimes \mathrm{BS}(n_2, m_2)$ are stably isomorphic, then*

- *$n_1 = n_2$ and $m_1 = m_2$, if $n_1 \neq |m_1|$;*
- *$n_1 = n_2$ and $|m_1| = |m_2|$, if $n_1 = |m_1|$.*

Proof. Define $M_i := L^\infty(X_i) \rtimes \mathrm{BS}(n_i, m_i)$ and $N_{i,z} := L^\infty(X_i) \rtimes \langle a_i^z \rangle$ for every nonzero integer z. Since the action $\mathrm{BS}(n_i, m_i) \curvearrowright (X_i, \mu_i)$ is essentially free, we have that $L^\infty(X_i)' \cap M_i = L^\infty(X_i)$. Therefore,

$$N_{i,z}' \cap M_i \subset L^\infty(X_i).$$

Hence, for every nonzero integer z, the relative commutant of $N_{i,z}$ inside M_i is equal to the algebra of $\langle a_i^k \rangle$-invariant functions in $L^\infty(X_i)$. By the aperiodicity of the action $\langle a_i \rangle \curvearrowright X_i$, we get that $N_{i,z}$ is an irreducible subalgebra of M_i for every z. Using Theorem C yields the desired result. $\square$

Remark. *It is important to note that whenever the crossed products arising in Corollary D would have a unique Cartan subalgebra up to unitary conjugacy, then Corollary D would immediately follow from the result of Houdayer and Raum using Singer's theorem (Theorem 1.3.13). To the best of our knowledge, there exist no such uniqueness results for these specific crossed products as of this writing. However, in Subsection 3.5, we present examples of essentially free ergodic pmp actions of Baumslag-Solitar groups where the corresponding crossed product has more than one Cartan subalgebra up to unitary conjugacy, but where $\langle a\rangle$ does not act ergodically. These examples were communicated to us by Anna Krogager.*

Let us give a short outline of the proof of Theorem C. For $i \in \{1,2\}$, set $\Gamma_i := \mathrm{BS}(n_i, m_i)$, $M_i := P_i \rtimes \Gamma_i$ and $N_i := P_i \rtimes \langle a_i\rangle$. Let p be a nonzero projection of N_1 and let $\alpha : pM_1p \to M_2$ be an isomorphism. The key to proving the main theorem is to show that $\alpha(pN_1p)$ and N_2 are unitarily conjugate.

Theorem 3.1.1 below will be playing a crucial role in proving this. It is our main technical result and is heavily inspired by Lemma 8.4 of [IPP08]. It roughly says that the relative commutant in $P \rtimes \mathrm{BS}(n,m)$ of every irreducible finite index subalgebra of $P \rtimes \langle a\rangle$ can be controlled in a good way. Concretely, set $M := P \rtimes \mathrm{BS}(n,m)$ and $N := P \rtimes \langle a\rangle$. If p is a nonzero projection of N and $Q \subset pNp$ is an irreducible finite index inclusion, then there exists a unitary $u \in \mathcal{U}(pMp)$ such that $uQu^* \subset pNp$ and $u(Q' \cap pMp)u^* \subset pNp$.

To obtain the unitary conjugacy of $\alpha(pN_1p)$ and N_2 we then start by showing that $N_2 \prec_{M_2} \alpha(pN_1p)$ and $\alpha(pN_1p) \prec_{M_2} N_2$. Let us only explain how to obtain the first intertwining, since the second intertwining can be obtained in a similar way. Using a slight adaptation of [BV12, Lemma 2.3], it actually suffices to show that

$$P_2 \prec_{M_2} \alpha(pN_1p) \tag{3.1}$$

and

$$L(\langle a_2\rangle) \prec_{M_2} \alpha(pN_1p). \tag{3.2}$$

Intertwining (3.1) is an immediate corollary of Theorem 4.1 from [Va13] on normalizers in HNN extensions of von Neumann algebras. For intertwining (3.2), we do the following. Let $\mathcal{C}$ be the centralizer of $\langle a_2^{n_2}\rangle < \Gamma_2$. By Lemma 1.11.2 and lemma 1.9.4 we have that $L(\mathcal{C})$ has no amenable direct summand. Then using Proposition 1.12.5, we see that $\alpha^{-1}(L(\mathcal{C})' \cap M_2) \prec pN_1p$. It follows that $\alpha^{-1}(L(\langle a_2^{n_2}\rangle)) \prec pN_1p$ and since $\langle a_2^{n_2}\rangle < \langle a_2\rangle$ has finite index, also $\alpha^{-1}(L(\langle a_2\rangle)) \prec pN_1p$. Applying α to both sides, we find intertwining (3.2).

After that, we use our main technical result to control relative commutants and show that the two-sided intertwining of the algebras $\alpha(pN_1p)$ and N_2 actually implies unitary conjugacy.

The rest of the proof of Theorem C can then be outlined as follows. By combining all of the above, we may assume that $\alpha(pN_1p) = N_2$. From this, we get that ${}_{N_2}L^2(M_2)_{N_2}$ is spanned by the irreducible bimodules $\overline{\text{span}}\ N_2u_gN_2$, but also by the irreducible bimodules $\overline{\text{span}}\ \alpha(pN_1u_gN_1p)$. By examining the left and right dimensions of these bimodules, we get that $\{(l(g), r(g)) \mid g \in \Gamma_1\}$ and $\{(l(g), r(g)) \mid g \in \Gamma_2\}$ coincide. This forces both $n_1 = n_2$ and $|m_1| = |m_2|$. When $n_1 \neq |m_1|$, a further careful study of the bimodules $\overline{\text{span}}\ N_2u_gN_2$ and $\overline{\text{span}}\ \alpha(pN_1u_gN_1p)$ moreover yields $m_1 = m_2$.

3.1 Controlling relative commutants

Fix integers $n, m \in \mathbb{Z}$ such that $2 \leq n \leq |m|$. Let (P, τ) be a diffuse tracial von Neumann algebra and let $\text{BS}(n, m) \curvearrowright P$ be a trace preserving action such that $P \rtimes \langle a^{l(g)} \rangle \subset P \rtimes \text{BS}(n, m)$ is irreducible for every $g \in \text{BS}(n, m)$. We write $\Gamma := \text{BS}(n, m)$, $M := P \rtimes \Gamma$, $N := P \rtimes \langle a \rangle$ and $N_z := P \rtimes \langle a^z \rangle$ for every nonzero integer z.

The following theorem is our main technical result and is, as we mentioned before, heavily inspired by Lemma 8.4 from [IPP08]. Roughly speaking, it lets us control relative commutants in M of irreducible finite index von Neumann subalgebras of N, allowing us later to deduce unitary conjugacy from a two-sided intertwining.

Theorem 3.1.1. *Let p be a nonzero projection of N. Let $Q \subset pNp$ be an irreducible finite index inclusion. Then there exists a unitary $u \in \mathcal{U}(pMp)$ such that $uQu^* \subset pNp$ and $u(Q' \cap pMp)u^* \subset pNp$.*

Proof. Since N is a II_1 factor and P is a diffuse von Neumann subalgebra, we may actually assume that the projection p from the description of the theorem is an element of P. So let p be a nonzero projection of P and let $Q \subset pNp$ be an irreducible finite index inclusion.

For every $g \in \Gamma$, we denote by $\mathcal{K}_g$ the N-N-subbimodule $\overline{\text{span}}\, Nu_gN$ of $L^2(M)$. Note that $\mathcal{K}_g$ is the closed linear span of $\{bu_h \mid b \in P,\ h \in \langle a \rangle g \langle a \rangle\}$. So $\mathcal{K}_g$ and $\mathcal{K}_h$ coincide if and only if $\langle a \rangle g \langle a \rangle = \langle a \rangle h \langle a \rangle$. Define $\hat{\Gamma} = \langle a \rangle \backslash \Gamma / \langle a \rangle$, i.e. the set of all double classes of $\langle a \rangle \leq \Gamma$. Then clearly

$$ {}_NL^2(M)_N = \bigoplus_{\langle a \rangle g \langle a \rangle \in \hat{\Gamma}} {}_N(\mathcal{K}_g)_N. $$

For every $g \in \Gamma$, we denote by p_g the orthogonal projection of $L^2(M)$ onto $\mathcal{K}_g$.

Whenever $x \in M$, we see that

$$p_g(x) = \sum_{i=0}^{l(g)-1} E_N(xu^*_{ga^i})u_{ga^i} \in M.$$

In particular, if $x \in Q' \cap pMp$, then $p_g(x)$ is a Q-central vector of $\mathcal{K}_g \cap pMp$. From this we get that $Q' \cap pMp$ is $||\cdot||_2$-norm densely spanned by the Q-central vectors of $\mathcal{K}_g \cap pMp$, where g runs over all elements of Γ. So investigating $Q' \cap pMp$ comes down to investigating the Q-central vectors of $\mathcal{K}_g \cap pMp$ for every $g \in \Gamma$. To that end, define $\Delta = \{g \in \Gamma \mid \mathcal{K}_g \cap pMp$ has a nonzero Q-central vector$\}$ and $\hat{\Delta} = \{\langle a\rangle g\langle a\rangle \in \hat{\Gamma} \mid g \in \Delta\}$.

Since $Q \subset pNp$ has finite index and $pNp \subset pMp$ is irreducible, we have that $Q' \cap pMp$ is finite dimensional (see e.g. [Va08, Lemma A.3]). On the other hand, different elements of $\hat{\Delta}$ give rise to orthogonal nonzero elements of $Q' \cap pMp$. Altogether $\hat{\Delta}$ is a finite set, say $\hat{\Delta} = \{\langle a\rangle g_1\langle a\rangle, \dots, \langle a\rangle g_\kappa\langle a\rangle\}$. We also have the following claim.

Claim 1. If $g \in \Delta$, then $l(g) = r(g)$.

Let $g \in \Delta \setminus \langle a\rangle$ and let x be a nonzero Q-central vector of $\mathcal{K}_g \cap pMp$. For every integer $z > 0$, we define

$$\mathcal{K}_g^{\otimes z} := \underbrace{\mathcal{K}_g \otimes_N \dots \otimes_N \mathcal{K}_g}_{z \text{ times}}.$$

We first show that $x^{\otimes z} \in (\mathcal{K}_g)^{\otimes z}$ is a nonzero element. Note that $E_N(x^*x)$ is an element of $Q' \cap pNp = \mathbb{C}p$. Hence $E_N(x^*x) = (||x||^2_{L^2(M)}/\tau(p))p = ||x||^2_{L^2(pMp)}p$. Therefore, for every integer $z > 0$, the element $x^{\otimes z} \in (\mathcal{K}_g)^{\otimes z}$ satisfies $||x^{\otimes z}|| = ||x||_{L^2(M)}||x||^{z-1}_{L^2(pMp)}$. In particular, $x^{\otimes z}$ is indeed a nonzero element of $(\mathcal{K}_g)^{\otimes z}$.

Now, for every integer $z > 0$, we define the nonzero pNp-pNp-subbimodule

$$\mathcal{H}_z := \overline{\text{span}}\, pN(x^{\otimes z})Np \subset p(\mathcal{K}_g^{\otimes z})p.$$

By Lemma 3.1.2, we have for every integer $z > 0$ that $\mathcal{H}_z$ is isomorphic with a pNp-pNp-subbimodule of $L^2(pNp) \otimes_Q L^2(pNp)$. By Proposition 1.7.3, this bimodule $L^2(pNp) \otimes_Q L^2(pNp)$ is finite index. Since pNp is also a factor, we find using Lemma 3.1.3 that $L^2(pNp) \otimes_Q L^2(pNp)$ only has a finite number of nonisomorphic pNp-pNp-subbimodules. Hence, there exist two nonzero positive integers z_1 and z_2 such that $z_1 \neq z_2$ and ${}_{pNp}(\mathcal{H}_{z_1})_{pNp} \cong {}_{pNp}(\mathcal{H}_{z_2})_{pNp}$.

Let us conclude the proof of Claim 1 by showing that for every integer $z > 0$ and every nonzero pNp-pNp-subbimodule $\mathcal{H}$ of $p(\mathcal{K}_g^{\otimes z})p$, we have

$$\dim_{pNp-}(\mathcal{H})/\dim_{-pNp}(\mathcal{H}) = (l(g)/r(g))^z.$$

So let $z > 0$ be an integer. We define $l = k(n_0^s m_0^t)^z$, where s and t satisfy $l(g) = kn_0^s m_0^t$. We define r analogously. For $0 \leq i_1, \dots, i_z < l(g)$, we introduce the pNp-pN_lp-subbimodule $\mathcal{H}_{(i_1,\dots,i_z)}$ of $p(\mathcal{K}_g^{\otimes z})p$ as

$${}_{pNp}(\mathcal{H}_{(i_1,\dots,i_z)})_{pN_lp} = {}_{pNp}(\overline{pN(u_{ga^{i_1}} \otimes \dots \otimes u_{ga^{i_z}})p})_{pN_lp}.$$

Since, for every projection $q \in P$ and every $0 \leq i, j < l(g)$,

$$E_N(u_{ga^i} q u_{ga^j}^*) = \begin{cases} u_{ga^i} q u_{ga^i}^* & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases}$$

we have that the bimodules $\mathcal{H}_{(i_1,\dots,i_z)}$ are pairwise orthogonal. Furthermore, since $\mathcal{K}_g$ is spanned by $L^2(N)u_g, \dots, L^2(N)u_{ga^{(l(g)-1)}}$ we have that $p(\mathcal{K}_g^{\otimes z})p$ is spanned by the bimodules $\mathcal{H}_{(i_1,\dots,i_z)}$. Altogether, we have found that

$${}_{pNp}(p(\mathcal{K}_g^{\otimes z})p)_{pN_lp} = \bigoplus_{(i_1,\dots,i_z)} {}_{pNp}(\mathcal{H}_{(i_1,\dots,i_z)})_{pN_lp}.$$

Fix $0 \leq i_1, \dots, i_z < l(g)$ and write β for $\mathrm{Ad}(u_{ga^{i_1}\dots ga^{i_z}})$ on pN_lp, then

$${}_{pNp}(\mathcal{H}_{(i_1,\dots,i_z)})_{pN_lp} \cong {}_{pNp}\mathcal{H}(\beta)_{pN_lp},$$

where $\mathcal{H}(\beta)$ is given by $\mathcal{H}(\beta) = pL^2(N)\beta(p)$ and $x\xi y = x\xi\beta(y)$. Since $\beta(pN_lp) = \beta(p)N_r\beta(p) \subset \beta(p)N\beta(p)$ is an irreducible inclusion, we have that the bimodule $\mathcal{H}(\beta)$ is irreducible. Also, its left dimension is 1 and its right dimension is r. We find from all of this that $p(\mathcal{K}_g^{\otimes z})p$ is orthogonally spanned by irreducible pNp-pN_lp-subbimodules having left dimension 1 and right dimension r. Since every nonzero pNp-pN_lp-subbimodule $\mathcal{K}$ of $p(\mathcal{K}_g^{\otimes z})p$ is a direct sum of irreducible pNp-pN_lp-subbimodules of $p(\mathcal{K}_g^{\otimes z})p$, we have that every nonzero pNp-pN_lp-subbimodule $\mathcal{K}$ of $p(\mathcal{K}_g^{\otimes z})p$ satisfies

$$\dim_{pNp-}(\mathcal{K})/\dim_{-pN_lp}(\mathcal{K}) = 1/r.$$

Now let $\mathcal{H}$ be a nonzero pNp-pNp-subbimodule of $p(\mathcal{K}_g^{\otimes z})p$. By Proposition 2.3.5 of [JS97], we have that $\dim_{-pN_lp}(\mathcal{H}) = [pNp : pN_lp]\dim_{-pNp}(\mathcal{H})$ and therefore

$$\begin{aligned} \dim_{pNp-}(\mathcal{H})/\dim_{-pNp}(\mathcal{H}) &= [pNp : pN_lp](\dim_{pNp-}(\mathcal{H})/\dim_{-pN_lp}(\mathcal{H})) \\ &= l/r = (l(g)/r(g))^z \end{aligned}$$

This concludes the proof of Claim 1.

We continue with the proof of the theorem. Take $\tilde{n} = \mathrm{lcm}(\{l(g) \mid g \in \Delta\})$. Note that $\tilde{n} \in \{l(g) \mid g \in \Gamma\}$. By Claim 1, we have that $u_g N_{\tilde{n}} u_g^* = N_{\tilde{n}}$ for every $g \in \Delta$. Let us now view Q inside an amplification of $N_{\tilde{n}}$ in the following sense. Since $Q \subset pNp$ and $N_{\tilde{n}} \subset N$ are finite index inclusions, there exists an integer $d > 0$, a projection $q \in N_{\tilde{n}}^d$, a normal $*$-homomorphism $\psi : Q \to qN_{\tilde{n}}^d q$ and a nonzero partial isometry $v \in (M_{1,d}(\mathbb{C}) \otimes pN)q$ such that

- $\psi(Q) \subset qN_{\tilde{n}}^d q$ is finite index;
- $v\psi(x) = xv$ for every $x \in Q$.

Since $\psi(Q) \subset qN_{\tilde{n}}^d q$ is finite index and $qN_{\tilde{n}}^d q$ is a factor, we have by [Va08, Lemma A.3] that $\psi(Q)' \cap qN_{\tilde{n}}^d q$ is finite dimensional. Cutting $qN_{\tilde{n}}^d q$ with a minimal projection of $\psi(Q)' \cap qN_{\tilde{n}}^d q$, we may actually assume that $\psi(Q) \subset qN_{\tilde{n}}^d q$ is irreducible.

Since $vv^* \in Q' \cap pNp = \mathbb{C}p$, we have that $vv^* = p$. Let us take a closer look at $v^*(Q' \cap pMp)v \subset qM^d q$. Recall that $\hat{\Delta} = \{\langle a \rangle g_1 \langle a \rangle, \dots, \langle a \rangle g_\kappa \langle a \rangle\}$. For $1 \leq \alpha \leq \kappa$, let x_α be a nonzero Q-central vector of $\mathcal{K}_{g_\alpha} \cap pMp$. Note that $v^* x_\alpha v$ is a nonzero $\psi(Q)$-central vector of $q(M_d(\mathbb{C}) \otimes \mathcal{K}_{g_\alpha})q \cap qM^d q$. Furthermore

$$q(M_d(\mathbb{C}) \otimes \mathcal{K}_{g_\alpha})q = \bigoplus_{i=0}^{\tilde{n}-1} \bigoplus_{j=0}^{l(g_\alpha)-1} q(L^2(N_{\tilde{n}}^d)(1 \otimes u_{a^i g_\alpha a^j}))q,$$

as $qN_{\tilde{n}}^d q$-$qN_{\tilde{n}}^d q$-bimodules. Since $Q' \cap pMp$ is spanned by the Q-central vectors of $\bigcup_{i=1}^{\kappa} (\mathcal{K}_{g_i} \cap pMp)$, we find from all this that

$$v^*(Q' \cap pMp)v \subset \mathrm{span}\{q(N_{\tilde{n}}^d(1 \otimes u_h))q \mid h \in \mathcal{L}\} \tag{3.3}$$

for some finite set $\mathcal{L} \subset \Gamma$. Moreover we see that $\mathcal{L}$ can be chosen to lie in Ω, where Ω equals

$$\{g \in \Gamma \mid u_g N_{\tilde{n}} u_g^* = N_{\tilde{n}} \text{ and } q(N_{\tilde{n}}^d(1 \otimes u_g))q \text{ has a nonzero } \psi(Q)\text{-central vector}\}.$$

Note that Ω is a group. Indeed, by the irreducibility of $\psi(Q) \subset qN_{\tilde{n}}^d q$ we have that Ω coincides with the group

$$\{g \in \Omega \mid q(N_{\tilde{n}}^d(1 \otimes u_g))q \text{ has a } \psi(Q)\text{-central vector } x \text{ with } xx^* = q = x^*x\}.$$

We also have the following claim.

Claim 2. For every finite subset $\mathcal{L} \subset \Omega$, there exists an element $g_0 \in \Gamma$ such that $r(g_0) \mid \tilde{n}$ and $\mathcal{L} \subset g_0 \langle a \rangle g_0^{-1}$.

We first show that every element of Ω is elliptic with respect to the action of Γ on its Bass-Serre tree T (see Section 1.11). So let g be an element of Ω. By definition of Ω, there exists a $\psi(Q)$-central vector x of $q(N^d_{\tilde{n}}(1 \otimes u_g))q$ satisfying $u_g N_{\tilde{n}} u_g^* = N_{\tilde{n}}$ and $xx^* = q = x^*x$. Note that $\mathrm{Ad}(x) \in \mathrm{Aut}(qN^d_{\tilde{n}}q)$ satisfies $\mathrm{Ad}(x)|_{\psi(Q)} = \mathrm{id}|_{\psi(Q)}$. Since $\psi(Q) \subset qN^d_{\tilde{n}}q$ is finite index and irreducible, the group of automorphisms of $qN^d_{\tilde{n}}q$ that restrict to the identity on $\psi(Q)$ is finite (see e.g. [Fa09, Lemma 8.12]). Therefore, there exists an integer $z > 0$ such that $\mathrm{Ad}(x)^z = \mathrm{id}$ on $qN^d_{\tilde{n}}q$. Equivalently $x^z \in (qN^d_{\tilde{n}}q)' \cap qM^dq = \mathbb{C}q$. On the other hand, x^z is a nonzero element of $q(N^d_{\tilde{n}}(1 \otimes u_{g^z}))q$. Therefore, g^z must be an element of $\langle a^{\tilde{n}} \rangle$ and hence g^z must be an elliptic element. Lemma 1.11.3.(1) implies that g must also be elliptic.

Now let $\mathcal{L} = \{g_1, \dots, g_t\}$ be a finite subset of Ω. Let $g, h \in \Omega$ be arbitrary elements. As we just showed, g and h are elliptic. Since Ω is a group, we have that $gh \in \Omega$ and hence also gh is elliptic. Using Lemma 1.11.3.(2), we get that the fixed point sets of g and h intersect nontrivially whenever $g, h \in \Omega$. Write $T_i = \{x \in V(T) \mid g_i \cdot x = x\}$ for $1 \leq i \leq t$. Then T_i is a subtree of T. Furthermore we already know that $T_i \cap T_j \neq \emptyset$ for every $1 \leq i, j \leq t$. It is an easy exercise to verify that finitely many subtrees of a given tree with pairwise nontrivial intersections have a nontrivial intersection. Hence there exists a vertex $x \in V(T)$ such that $g_i \cdot x = x$ for every i. In other words, there exists an element $g \in \Gamma$ such that $g_i \in g\langle a \rangle g^{-1}$ for every i. Define $\mathcal{J} = \{g \in \Gamma \mid \mathcal{L} \subset g\langle a \rangle g^{-1}\}$. We already showed that $\mathcal{J}$ is nonempty. Choose an element $g_0 \in \mathcal{J}$ having minimal b-length (see Section 1.11). Using Lemma 1.11.1 and the fact that $r(g) \mid \tilde{n}$ for every $g \in \Omega$, it follows that $r(g_0) \mid \tilde{n}$. This concludes the proof of Claim 2.

Let us now finish the proof of the theorem. Combining inclusion (3.3) with Claim 2, we find that there exists an element $g_0 \in \Gamma$ such that $r(g_0) \mid \tilde{n}$ and $v^*(Q' \cap pMp)v \subset \mathrm{span}\{q(N^d_{\tilde{n}}(1 \otimes u_h))q \mid h \in g_0\langle a \rangle g_0^{-1}\}$.

Set $\tilde{v} = v(1 \otimes u_{g_0})$. Then

$$\begin{aligned}
\tilde{v}^*(Q' \cap pMp)\tilde{v} &= (1 \otimes u_{g_0}^*)v^*(Q' \cap pMp)v(1 \otimes u_{g_0}) \\
&\subset \mathrm{span}\{(1 \otimes u_{g_0}^*)(N^d_{\tilde{n}}(1 \otimes u_h))(1 \otimes u_{g_0}) \mid h \in g_0\langle a \rangle g_0^{-1}\} \\
&\subset \mathrm{span}\{(N^d(1 \otimes u_{g_0^{-1}hg_0}) \mid h \in g_0\langle a \rangle g_0^{-1}\} \\
&= N^d.
\end{aligned}$$

In particular $\tilde{v}^*\tilde{v} \in N^d$.

Furthermore we have that

$$\begin{aligned}\tilde{v}^*Q\tilde{v} &= \tilde{v}^*\tilde{v}(1\otimes u_{g_0}^*)\psi(Q)(1\otimes u_{g_0})\\ &\subset N^d(1\otimes u_{g_0}^*)N_{\tilde{n}}^d(1\otimes u_{g_0})\\ &= N^d.\end{aligned}$$

Let w be an element of $M_{d,1}(\mathbb{C})\otimes N$ such that $ww^* = \tilde{v}^*\tilde{v}$ and $w^*w = \tilde{v}\tilde{v}^* = p$. Define $u = (\tilde{v}w)^* \in pMp$. Then $uu^* = w^*\tilde{v}^*\tilde{v}w = w^*ww^*w = w^*w = p$ and $u^*u = \tilde{v}ww^*\tilde{v}^* = \tilde{v}\tilde{v}^*\tilde{v}\tilde{v}^* = \tilde{v}\tilde{v}^* = p$. So $u \in \mathcal{U}(pMp)$. Also

$$\begin{aligned}uQu^* &= w^*\tilde{v}^*Q\tilde{v}w\\ &\subset w^*N^dw\\ &\subset N\end{aligned}$$

and

$$\begin{aligned}u(Q'\cap pMp)u^* &= w^*\tilde{v}^*(Q'\cap pMp)\tilde{v}w\\ &\subset w^*N^dw\\ &\subset N.\end{aligned}$$

This ends the proof. □

In the proof of Theorem 3.1.1, we used the following two well known results.

Lemma 3.1.2. *Let (M,τ) be a II_1 factor and let $N\subset M$ be an irreducible, finite index subfactor. Let ${}_M\mathcal{H}_M$ be an M-M-bimodule and let $\xi\in\mathcal{H}$ be a nonzero N-central vector such that* span $M\xi M$ *is dense in $\mathcal{H}$. Then $\mathcal{H}$ is isomorphic with an M-M-subbimodule of $L^2(M)\otimes_N L^2(M)$.*

Proof. By replacing ξ with $\xi/||\xi||$, we may assume that $\xi\in\mathcal{H}$ is a unit vector. Let us begin by showing that $L^2(M)$ and $\overline{\xi M}$ are isomorphic as N-M-bimodules. For that define $\varphi : M\to\mathbb{C} : x\mapsto\langle\xi x,\xi\rangle$. Then φ is a normal N-central state on M. Since $N\subset M$ is irreducible, this implies that φ and τ coincide, see e.g. [Tak03, Lemma 2.12.(iii)]. Hence

$$||\xi x||^2 = \langle\xi x,\xi x\rangle = \varphi(xx^*) = \tau(xx^*) = ||x||_2^2.$$

From this we find that we can extend the map $\alpha : M\to\overline{\xi M} : x\mapsto\xi x$ to a unitary from $L^2(M)$ onto $\overline{\xi M}$. This unitary is N-M-bimodular by construction.

Let us continue with the proof of the lemma. Use Lemma 2.8 from [FR12] where ${}_M\mathcal{K}_N := {}_M L^2(M)_N$ and ${}_N\mathcal{L}_M = {}_N\overline{\xi M}_M \subset {}_N L^2(M)_M$. Then we get that $\mathcal{H}$ is an M-M-bimodule isomorphic with a subbimodule of $L^2(M) \otimes_N \overline{\xi M}$. Since we already showed that $L^2(M)$ and $\overline{\xi M}$ are isomorphic as N-M-bimodules, we are done. $\square$

Lemma 3.1.3. *Let (M, τ) be a II_1 factor and let ${}_M\mathcal{H}_M$ be a nonzero finite index bimodule. Then $\mathcal{H}$ only contains a finite number of nonisomorphic M-M-subbimodules.*

Proof. Let (M, τ) be a II_1 factor and let ${}_M\mathcal{H}_M$ be a finite index bimodule. As before, there exists a nonzero positive integer n, a nonzero projection $p \in M^n$ and a normal $*$-homomorphism $\psi : M \to pM^n p$ such that $[pM^n p : \psi(M)] < \infty$ and

$${}_M\mathcal{H}_M \cong {}_M\mathcal{H}(\psi)_M,$$

where ${}_M\mathcal{H}(\psi)_M$ is given by $\mathcal{H}(\psi) = p(\mathbb{C}^n \otimes L^2(M))$ and $x\xi y = \psi(x)\xi y$. In this way, we see that the isomorphism classes of the M-M-subbimodules of $\mathcal{H}$ correspond with the equivalence classes of the projections in $\psi(M)' \cap pM^n p$. On the other hand, $\psi(M)' \cap pM^n p$ is finite dimensional by [Va08, Lemma A.3]. Hence

$$\{(\mathrm{Tr}_{M_n(\mathbb{C})} \otimes \tau)(q) \mid q \in \psi(M)' \cap pM^n p \text{ is a projection}\}$$

is a finite set. To end the proof, Proposition 1.1.2 of [JS97] states that $\mathrm{Tr}_{M_n(\mathbb{C})} \otimes \tau$ is a complete invariant for the equivalence classes of projections in M^n, since M^n is a factor. $\square$

3.2 Unitary conjugacy of the canonical subalgebras $\alpha(pN_1p)$ and N_2

Fix integers $n_1, m_1, n_2, m_2 \in \mathbb{Z}$ satisfying $2 \leq n_1 \leq |m_1|$ and $2 \leq n_2 \leq |m_2|$. For $i \in \{1, 2\}$, let (P_i, τ_i) be a diffuse amenable tracial von Neumann algebra and let $\mathrm{BS}(n_i, m_i) \curvearrowright P_i$ be a trace preserving action such that $P_i \rtimes \langle a_i^{l(g)} \rangle \subset P_i \rtimes \mathrm{BS}(n_i, m_i)$ is irreducible for every $g \in \mathrm{BS}(n_i, m_i)$. We still denote by τ_i the canonical trace of $P_i \rtimes \mathrm{BS}(n_i, m_i)$.

We make use of the following notation: $\Gamma_i = \mathrm{BS}(n_i, m_i)$, $M_i = P_i \rtimes \Gamma_i$, $N_i = P_i \rtimes \langle a_i \rangle$ and $N_{i,z} = P_i \rtimes \langle a_i^z \rangle$.

Proposition 3.2.1. *Let p be a nonzero projection of N_1. If $\alpha : pM_1p \to M_2$ is an isomorphism, then there exists a nonzero irreducible finite index $\alpha(pN_1p)$-N_2-subbimodule of $L^2(M_2)$.*

Proof. Since P_1 is diffuse and N_1 is a II_1 factor, we may actually assume that p is a nonzero projection of P_1. So let p be a nonzero projection of P_1 and let $\alpha : pM_1p \to M_2$ be an isomorphism.

We first prove that $N_2 \prec \alpha(pN_1p)$. Denote by $\mathcal{C}$ the centralizer of $\langle a_2^{n_2}\rangle$ inside Γ_2. Recall that by Lemma 1.11.2 and Lemma 1.9.4 the group von Neumann algebra $L(\mathcal{C})$ has no amenable direct summand. Also recall that, by Lemma 1.12.3, we have $M_1 = \mathrm{HNN}(N_1, N_{1,n_1}, \mathrm{Ad}(u_{b_1}))$. Using Proposition 1.12.5 on $\alpha^{-1}(L(\mathcal{C})) \subset pM_1p$ we get that

$$\alpha^{-1}(L(\mathcal{C}))' \cap pM_1p \prec N_1.$$

Since $\alpha^{-1}(L(\langle a_2^{n_2}\rangle))$ is a subalgebra of $\alpha^{-1}(L(\mathcal{C}))' \cap pM_1p$ this implies that $\alpha^{-1}(L(\langle a_2^{n_2}\rangle)) \prec N_1$. Combining that with Lemma 1.10.3 and the fact that $\alpha^{-1}(L(\langle a_2^{n_2}\rangle)) \subset \alpha^{-1}(L(\langle a_2\rangle))$ is a finite index inclusion, we get that $\alpha^{-1}(L(\langle a_2\rangle)) \prec N_1$. On the other hand, using Theorem 4.1 from [Va13], we find that $\alpha^{-1}(P_2) \prec N_1$. In fact, by Lemma 1.10.8, we even have that $\alpha^{-1}(P_2) \prec^f N_1$, since $P_2 \subset M_2$ is a regular inclusion. Applying Lemma 3.2.2 to $\alpha^{-1}(L(\langle a_2\rangle)) \prec N_1$ and $\alpha^{-1}(P_2) \prec^f N_1$, we obtain that $\alpha^{-1}(N_2) \prec_{M_1} N_1$. Since N_1 is a factor, this implies that $\alpha^{-1}(N_2) \prec_{M_1} pN_1p$ or equivalently $N_2 \prec_{M_2} \alpha(pN_1p)$.

We also show that $\alpha(pN_1p) \prec_{M_2} N_2$. Fix $n \in \mathbb{N}$ such that $n \geq 1/\tau_1(p)$ and choose a projection $q \in P_1$ such that $q \leq p$ and $\tau_1(q) = 1/n$. Define the isomorphism $\beta : (\alpha(q)M_2\alpha(q))^n \to (qM_1q)^n$ given by $1 \otimes \alpha^{-1}$. Let $v \in M_{1,n}(\mathbb{C}) \otimes P_1$ satisfy $vv^* = 1$ and $v^*v = 1 \otimes q$. Then $\mathrm{Ad}(v) : (qM_1q)^n \to M_1$ is an isomorphism. Now define the isomorphism $\gamma : (\alpha(q)M_2\alpha(q))^n \to M_1$ given by $\mathrm{Ad}(v) \circ \beta$. Using the exact same arguments as before, we get that $\gamma^{-1}(N_1) \prec_{M_2^n} N_2^n$ or equivalently that $(\alpha(qN_1q))^n \prec_{M_2^n} N_2^n$. By Lemma 1.10.4, this shows that $\alpha(pN_1p) \prec_{M_2} N_2$.

So far, we have found that $N_2 \prec_{M_2} \alpha(pN_1p)$ and $\alpha(pN_1p) \prec_{M_2} N_2$. Since $pN_1p \subset pM_1p$ and $N_2 \subset M_2$ are quasi-regular inclusions, a combination of Proposition 1.10.9 and Lemma 1.10.8 yields the desired result. $\square$

In the proof of Proposition 3.2.1 we used the following lemma which is a slight adaptation of Lemma 2.3 from [BV12].

Lemma 3.2.2. *Let Γ be a countable group and $\Gamma \curvearrowright (P, \tau)$ a trace preserving action of Γ on a tracial von Neumann algebra (P, τ). Put $M = P \rtimes \Gamma$ and let $p \in M$ be a projection. Assume that $Q \subset pMp$ is a von Neumann subalgebra that is normalized by a group of unitaries $\mathcal{G} \subset \mathcal{U}(pMp)$. Let $\Lambda \leq \Gamma$ be an almost normal subgroup. If $Q \prec_M^f P \rtimes \Lambda$ and $\mathcal{G}'' \prec_M P \rtimes \Lambda$, then $(Q \cup \mathcal{G})'' \prec_M P \rtimes \Lambda$.*

Proof. For every subset $\mathcal{F} \subset \Gamma$, we denote by $P_{\mathcal{F}}$ the orthogonal projection of $L^2(M)$ onto the closed linear span of $\{au_g \mid a \in P, g \in \mathcal{F}\}$. We say that a subset $\mathcal{F} \subset \Gamma$ is small relative to Λ if $\mathcal{F}$ is contained in a finite union of subsets of the form $g\Lambda h$ with $g, h \in \Gamma$.

Assume, by way of reaching a contradiction, that $(Q \cup \mathcal{G})'' \not\prec P \rtimes \Lambda$. Since $\mathcal{U}(Q)\mathcal{G}$ is a group of unitaries generating $(Q \cup \mathcal{G})''$, we get from [Va10, Lemma 2.4] sequences of unitaries $b_n \in U(Q)$ and $w_n \in \mathcal{G}$ such that $||P_{\mathcal{F}}(b_n w_n)||_2 \to 0$ for every subset $\mathcal{F} \subset \Gamma$ that is small relative to Λ.

Since $\mathcal{G}'' \prec P \rtimes \Lambda$, there exists a nonzero projection $q \in (P \rtimes \Lambda)^n$, a nonzero partial isometry $v \in M_{1,n}(\mathbb{C}) \otimes pM$ and a normal $*$-homomorphism $\psi : \mathcal{G}'' \to q(P \rtimes \Lambda)^n q$ such that $xv = v\psi(x)$ for all $x \in \mathcal{G}''$. Denote $p_1 = vv^*$ and fix $0 < \varepsilon < ||p_1||_2/3$. By the Kaplansky density theorem, we can take a finite subset $\mathcal{F}_1 \subset \Gamma$ and an element v_1 in the linear span of $\{au_g \mid a \in M_{1,n}(\mathbb{C}) \otimes P, g \in \mathcal{F}_1\}$ such that $||v_1|| \leq 1$ and $||v - v_1||_2 < \varepsilon$.

Denote $\mathcal{F}_2 = \mathcal{F}_1 \Lambda \mathcal{F}_1^{-1}$. Observe that $\mathcal{F}_2$ is small relative to Λ. Write $x_n = v_1 \psi(w_n) v_1^*$. By construction, every x_n lies in the image of $P_{\mathcal{F}_2}$. We also have for all n that $||x_n|| \leq 1$ and

$$\begin{aligned}
||w_n p_1 - x_n||_2 &= ||v\psi(w_n)v^* - v_1\psi(w_n)v_1^*||_2 \\
&\leq ||v\psi(w_n)v^* - v\psi(w_n)v_1^*||_2 + ||v\psi(w_n)v_1^* - v_1\psi(w_n)v_1^*||_2 \\
&\leq ||v\psi(w_n)||\ ||v^* - v_1^*||_2 + ||\psi(w_n)v_1^*||\ ||v - v_1||_2 \\
&= ||v||\ ||v^* - v_1^*||_2 + ||v_1||\ ||v - v_1||_2 \\
&< 2\varepsilon
\end{aligned}$$

Since $Q \prec^f P \rtimes \Lambda$, we obtain from [Va10, Lemma 2.5] a subset $\mathcal{F}_3 \subset \Gamma$ that is small relative to Λ such that $||b_n - P_{\mathcal{F}_3}(b_n)||_2 < \varepsilon$ for all n. In combination with the previous paragraph, we get that

$$\begin{aligned}
||b_n w_n p_1 - P_{\mathcal{F}_3}(b_n)x_n||_2 &\leq ||b_n w_n p_1 - b_n x_n||_2 + ||b_n x_n - P_{\mathcal{F}_3}(b_n)x_n||_2 \\
&\leq ||w_n p_1 - x_n||_2 + ||x_n||\ ||b_n - P_{\mathcal{F}_3}(b_n)||_2 \\
&< 3\varepsilon,
\end{aligned}$$

for all n. Denote $\mathcal{F}_4 = \mathcal{F}_3\mathcal{F}_2$. Since Λ is an almost normal subgroup of Γ, we have that $\mathcal{F}_4$ is still small relative to Λ. By construction, $P_{\mathcal{F}_3}(b_n)x_n$ lies in the image of $P_{\mathcal{F}_4}$ and thus we have shown that $||b_n w_n p_1 - P_{\mathcal{F}_4}(b_n w_n p_1)||_2 < 3\varepsilon$ for all n.

Since $||P_{\mathcal{F}}(b_n w_n)||_2 \to 0$ for every subset $\mathcal{F} \subset \Gamma$ that is small relative to Λ, it follows from [Va10, Lemma 2.3] that $||P_{\mathcal{F}_4}(b_n w_n p_1)||_2 \to 0$. Hence $\limsup_n ||b_n w_n p_1||_2 \leq 3\varepsilon$. Since b_n and w_n are unitaries, we arrive at the contradiction that $||p_1||_2 \leq 3\varepsilon < ||p_1||_2$. □

The next theorem states that the intertwining bimodule from Proposition 3.2.1 can actually be chosen to realize a unitary conjugacy.

Theorem 3.2.3. *Let p be a nonzero projection of N_1. If $\alpha : pM_1p \to M_2$ is an isomorphism, then $\alpha(pN_1p)$ and N_2 are unitarily conjugate in M_2.*

Proof. Let p be a nonzero projection of N_1 and let $\alpha : pM_1p \to M_2$ be an isomorphism. Then by Proposition 3.2.1, there exists an integer $d > 0$, a projection $q \in N_2^d$, a normal $*$-homomorphism $\psi : \alpha(pN_1p) \to qN_2^dq$ and a nonzero partial isometry $v \in (M_{1,d}(\mathbb{C}) \otimes M_2)q$ such that

- $\psi(\alpha(pN_1p)) \subset qN_2^dq$ is irreducible and finite index;
- $v\psi(x) = xv$ for every $x \in \alpha(pN_1p)$.

Identifying N_2^d with $P_2^d \rtimes \langle a_2 \rangle$ and identifying M_2^d with $P_2^d \rtimes \Gamma_2$, we can apply Theorem 3.1.1 to $\psi(\alpha(pN_1p)) \subset qN_2^dq \subset qM_2^dq$. This yields a unitary $u \in \mathcal{U}(qM_2^dq)$ such that

$$u\psi(\alpha(pN_1p))u^* \subset qN_2^dq \text{ and } u(\psi(\alpha(pN_1p))' \cap qM_2^dq)u^* \subset qN_2^dq.$$

Now define $\tilde{\psi} = \mathrm{Ad}(u) \circ \psi$ and $\tilde{v} = vu^*$. Then

- $\tilde{\psi} : \alpha(pN_1p) \to qN_2^dq$ is a normal $*$-homomorphism;
- $\tilde{v}\tilde{\psi}(x) = x\tilde{v}$ for every $x \in \alpha(pN_1p)$.

Note that $\tilde{v}\tilde{v}^*$ is a nonzero projection of $\alpha(pN_1p)' \cap M_2 = \mathbb{C}1$. Hence $\tilde{v}\tilde{v}^*$ must be equal to 1. On the other hand, $\tilde{v}^*\tilde{v} = uv^*vu^* \in u(\psi(\alpha(pN_1p))' \cap qM_2^dq)u^* \subset qN_2^dq$. Let w be an element of $M_{d,1}(\mathbb{C}) \otimes N_2$ such that $ww^* = \tilde{v}^*\tilde{v}$ and $w^*w = \tilde{v}\tilde{v}^* = 1$. Then $u_1 = (\tilde{v}w)^* \in M_2$. Furthermore $u_1u_1^* = w^*\tilde{v}^*\tilde{v}w = w^*ww^*w = 1$ and $u_1^*u_1 = \tilde{v}ww^*\tilde{v}^* = \tilde{v}\tilde{v}^*\tilde{v}\tilde{v}^* = 1$. So $u_1 \in \mathcal{U}(M_2)$. Also

$$\begin{aligned} u_1\alpha(pN_1p)u_1^* &= w^*\tilde{v}^*\alpha(pN_1p)\tilde{v}w = w^*\tilde{v}^*\tilde{v}\tilde{\psi}(\alpha(pN_1p))w \\ &= w^*\tilde{\psi}(\alpha(pN_1p))w \subset w^*N_2^dw \\ &\subset N_2. \end{aligned}$$

By exactly the same arguments, there also exists a unitary u_2 of pM_1p such that $u_2\alpha^{-1}(N_2)u_2^* \subset pN_1p$. Applying α to both sides, there exists a unitary u_3 of M_2 satisfying $u_3N_2u_3^* \subset \alpha(pN_1p)$.

Combining both inclusions, we get that $u_1u_3N_2u_3^*u_1^* \subset u_1\alpha(pN_1p)u_1^* \subset N_2$. To finish the proof, it suffices to show that $u_1u_3 \in N_2$. To that end, denote the unitary $u_1u_3 \in M_2$ by u_4 and write β for $\mathrm{Ad}(u_4)$ on N_2. As before, define $\hat{\Gamma}_2 = \langle a\rangle\backslash\Gamma_2/\langle a\rangle$ and ${}_{N_2}(\mathcal{K}_g)_{N_2} = {}_{N_2}(\overline{\mathrm{span}}^{||\cdot||_2}N_2u_gN_2)_{N_2}$. Recall that ${}_{N_2}L^2(M_2)_{N_2} = \bigoplus_{\langle a\rangle g\langle a\rangle} {}_{N_2}(\mathcal{K}_g)_{N_2}$. For every $g \in \Gamma_2$, we define p_g to be the orthogonal projection of $L^2(M_2)$ onto $\mathcal{K}_g$. Then we can decompose u_4 as

$$u_4 = \sum_{\langle a\rangle g\langle a\rangle \in \hat{\Gamma}_2} p_g(u_4),$$

where the convergence is in $||\cdot||_2$-norm. Note that $p_g(u_4) = \sum_{i=0}^{l(g)-1} x_{g,i}u_{ga^i}$, where $x_{g,i} = E_{N_2}(u_4u_{ga^i}^*)$. Since $\beta(x)u_4 = u_4x$ for every $x \in N_2$, we get that

$$\beta(x)\Big(\sum_{i=0}^{l(g)-1} x_{g,i}u_{ga^i}\Big) = \Big(\sum_{i=0}^{l(g)-1} x_{g,i}u_{ga^i}\Big)x \text{ for every } g \in \Gamma_2 \text{ and every } x \in N_2.$$

But then, for every $g \in \Gamma_2$ and every $x \in N_{2,l(g)}$, we have that $\beta(x)(x_{g,i}u_{ga^i}) = (x_{g,i}u_{ga^i})x$. Therefore $(x_{g,i}u_{ga^i})^*(x_{g,i}u_{ga^i}) \in N_{2,l(g)}' \cap M_2$. By the irreducibility of $N_{2,l(g)} \subset M_2$, we get that $x_{g,i}^*x_{g,i} = x_{g,i}x_{g,i}^* \in \mathbb{C}1$. So $x_{g,i}$ is a multiple of a unitary for every $g \in \Gamma_2$ and every $0 \leq i < l(g)$.

Now assume that $x_{g,i}$ and $x_{h,j}$ are both nonzero. Note that $(x_{h,j}u_{ha^j})^*(x_{g,i}u_{ga^i})$ is an element of $N_{2,l}' \cap M_2$, where $l = \mathrm{lcm}(l(g), l(h))$. Since $N_{2,l}$ is irreducible in M_2 and $x_{h,j}^*x_{g,i}$ is nonzero, we get that $u_{ha^j} \in N_2u_{ga^i}$ and hence that $ha^j \in \langle a\rangle ga^i$. Therefore, in the decomposition of u_4, there is only one nonzero component $x_{g,i}u_{ga^i}$. Hence $u_4 = xu_g$ for some $g \in \Gamma_2$ and $x \in \mathcal{U}(N_2)$. But since $u_4N_2u_4^* \subset N_2$, we must have that $g \in \langle a\rangle$ and hence that $u_4 \in N_2$. This ends the proof. □

3.3 Proof of the main theorem

We begin with the following result.

Lemma 3.3.1. *Let $n, m \in \mathbb{Z}$ such that $2 \leq n \leq |m|$. Let (P, τ) be a tracial von Neumann algebra and let $\mathrm{BS}(n,m) \curvearrowright P$ be a trace preserving action such that $P \rtimes \langle a^{l(g)}\rangle \subset P \rtimes \mathrm{BS}(n,m)$ is irreducible for every $g \in \mathrm{BS}(n,m)$. Write $M := P \rtimes \mathrm{BS}(n,m)$, $N := P \rtimes \langle a\rangle$ and ${}_N(\mathcal{K}_g)_N := {}_N(\overline{\mathrm{span}}^{||\cdot||_2}Nu_gN)_N$. Then*

- ${}_N(\mathcal{K}_g)_N$ *is irreducible for every* $g \in \mathrm{BS}(n,m)$*;*
- ${}_N(\mathcal{K}_g)_N \cong {}_N(\mathcal{K}_h)_N$ *if and only if* $\langle a\rangle g\langle a\rangle = \langle a\rangle h\langle a\rangle$.

Proof. As before, we define $N_z := P \rtimes \langle a^z\rangle$ for every nonzero integer z. For every $g \in \mathrm{BS}(n,m)$ we have that

$$ {}_{N_{r(g)}}(\mathcal{K}_g)_N = \bigoplus_{i=0}^{r(g)-1} {}_{N_{r(g)}}(u_{a^i g} L^2(N))_N \,. \tag{3.4} $$

Let $g, h \in \mathrm{BS}(n,m)$ such that $r(g) = r(h)$. Under the identification (3.4) we have that the set of $N_{r(g)}$-N-bimodular elements of $B(\mathcal{K}_g, \mathcal{K}_h)$ coincides with

$$ B_{g,h} := \{[x_{i,j}]_{0\le i,j<r(g)} \mid x_{i,j} \in N'_{r(g)} \cap u_{a^i h} N u^*_{a^j g}\}. $$

Note that $N'_{r(g)} \cap u_{a^i h} N u^*_{a^j g}$ is a subset of $N'_{r(g)} \cap M = \mathbb{C}1$. Therefore we have that

$$ N'_{r(g)} \cap u_{a^i h} N u^*_{a^j g} = \mathbb{C}1 \cap u_{a^i h} N u^*_{a^j g} \text{ for every } 0 \le i,j < r(g). $$

Hence,

$$ B_{g,h} = \{[x_{i,j}]_{0\le i,j<r(g)} \mid x_{i,j} \in \mathbb{C}1 \cap u_{a^i h} N u^*_{a^j g}\}. $$

From this, it follows that $B_{g,g} = \{[x_{i,j}]_{0\le i,j<r(g)} \mid x_{i,j} \in \mathbb{C}\delta_{i,j}\}$ and that $B_{g,h} = \{0\}$ whenever $\langle a\rangle g\langle a\rangle \neq \langle a\rangle h\langle a\rangle$.

Let us now prove the first statement. Assume that $\mathcal{H}$ is a nonzero N-N-subbimodule of $\mathcal{K}_g$. Denote by $p_{\mathcal{H}}$ the orthogonal projection of $\mathcal{K}_g$ onto $\mathcal{H}$. Then $p_{\mathcal{H}}$ is a nonzero element of $B_{g,g} = \{[x_{i,j}]_{0\le i,j<r(g)} \mid x_{i,j} \in \mathbb{C}\delta_{i,j}\}$. This implies that $\mathcal{H}$ contains $u_{a^i g}L^2(N)$ for some $0 \le i < r(g)$. Since $\mathcal{H}$ is also a left N-module, we see that $\mathcal{H}$ coincides with the whole of $\mathcal{K}_g$. This proves the first statement.

To prove the second statement, assume that ${}_N(\mathcal{K}_g)_N \cong {}_N(\mathcal{K}_h)_N$. Then, by comparing the right dimensions of both bimodules, we have that $r(g) = r(h)$. Furthermore, the unitary between $\mathcal{K}_g$ and $\mathcal{K}_h$ is an element of $B_{g,h}$. This implies that $B_{g,h} \neq \{0\}$, and hence that $\langle a\rangle g\langle a\rangle = \langle a\rangle h\langle a\rangle$. This ends also the proof of the second statement. $\square$

For the following lemma, we need to introduce some extra notation. Let $\omega \in \mathbb{C}$ satisfy $|\omega| = 1$. Let (P,τ) be a tracial von Neumann algebra and let $\mathbb{Z} \curvearrowright P$ be a trace preserving action. Write $N := P \rtimes \mathbb{Z}$. Then we define the $*$-automorphism $\alpha_\omega : N \to N$ by

$$ \alpha_\omega(bu_z) = \omega^z b u_z, $$

for every $b \in P$ and $z \in \mathbb{Z}$. Furthermore, we define ${}_N(\mathcal{K}_\omega)_N$ by $\mathcal{K}_\omega = L^2(N)$ and $x\xi y = \alpha_\omega(x)\xi y$.

Lemma 3.3.2. *Let $n, m \in \mathbb{Z}$ such that $2 \leq n \leq |m|$. Let (P, τ) be a tracial von Neumann algebra and let $\mathrm{BS}(n,m) \curvearrowright P$ be a trace preserving action such that $P \rtimes \langle a^{l(g)} \rangle \subset P \rtimes \mathrm{BS}(n,m)$ is irreducible for every $g \in \mathrm{BS}(n,m)$. Write $M := P \rtimes \mathrm{BS}(n,m)$, $N := P \rtimes \langle a \rangle$, $\omega_g := e^{2\pi i/r(g)}$ and ${}_N(\mathcal{K}_g)_N := {}_N(\overline{\mathrm{span}}^{\|\cdot\|_2} N u_g N)_N$. Then*

$$ {}_N(\mathcal{K}_g \otimes_N \mathcal{K}_{g^{-1}})_N \cong \left(\bigoplus_{i=0}^{r(g)-1} {}_N(\mathcal{K}_{\omega_g^i})_N \right) \oplus \left(\bigoplus_{i=1}^{l(g)-1} {}_N(\mathcal{K}_{ga^ig^{-1}})_N \right). \tag{3.5} $$

Moreover, the bimodules in the decomposition are irreducible and pairwise nonisomorphic.

Proof. To prove (3.5) we first show that

$$ {}_N(\mathcal{K}_g \otimes_N \mathcal{K}_{g^{-1}})_N = \bigoplus_{i=0}^{l(g)-1} {}_N\overline{\mathrm{span}}(N u_{ga^i} \otimes u_{g^{-1}} N)_N. $$

After that we show that ${}_N\overline{\mathrm{span}}(N u_{ga^i} \otimes u_{g^{-1}} N)_N \cong {}_N(\mathcal{K}_{ga^ig^{-1}})_N$ when $i \neq 0$ and $\overline{\mathrm{span}}(N u_g \otimes u_{g^{-1}} N) \cong \bigoplus_{i=0}^{r(g)-1} {}_N(\mathcal{K}_{\omega_g^i})_N$.

Let us begin. First of all, we have that $\mathcal{K}_g \otimes_N \mathcal{K}_{g^{-1}}$ is linearly spanned by the N-N-subbimodules $\overline{\mathrm{span}}(N u_{ga^i} \otimes u_{g^{-1}} N)$, where $0 \leq i < l(g)$. Furthermore, we have for $x, y, z, w \in N$ and $0 \leq i, j < l(g)$ that

$$ \begin{aligned} \langle x u_{ga^i} \otimes u_{g^{-1}} y, z u_{ga^j} \otimes u_{g^{-1}} w \rangle &= \langle x u_{ga^i} E_N(u_{g^{-1}} y w^* u_g), z u_{ga^j} \rangle \\ &= \langle x u_{ga^i} u_{g^{-1}} E_{N_{r(g)}}(y w^*) u_g, z u_{ga^j} \rangle \\ &= \langle x u_{ga^ig^{-1}} E_{N_{r(g)}}(y w^*), z u_{ga^jg^{-1}} \rangle. \end{aligned} \tag{3.6} $$

If $i \neq j$, then (3.6) implies that $\overline{\mathrm{span}}(N u_{ga^i} \otimes u_{g^{-1}} N)$ and $\overline{\mathrm{span}}(N u_{ga^j} \otimes u_{g^{-1}} N)$ are orthogonal. Hence,

$$ {}_N(\mathcal{K}_g \otimes_N \mathcal{K}_{g^{-1}})_N = \bigoplus_{i=0}^{l(g)-1} {}_N\overline{\mathrm{span}}(N u_{ga^i} \otimes u_{g^{-1}} N)_N. $$

If $i = j \neq 0$, then we can continue with (3.6) in the following way:

$$ \begin{aligned} \langle x u_{ga^ig^{-1}} E_{N_{r(g)}}(y w^*), z u_{ga^ig^{-1}} \rangle &= \langle x E_N(u_{ga^ig^{-1}} y w^* u_{ga^ig^{-1}}^*), z \rangle \\ &= \langle x u_{ga^ig^{-1}} y w^* u_{ga^ig^{-1}}^*, z \rangle \\ &= \langle x u_{ga^ig^{-1}} y, z u_{ga^ig^{-1}} w \rangle. \end{aligned} $$

This shows that ${}_N\overline{\text{span}}(Nu_{ga^i} \otimes u_{g^{-1}}N)_N \cong {}_N(\mathcal{K}_{ga^ig^{-1}})_N$ when $i \neq 0$. Hence,

$$ {}_N(\mathcal{K}_g \otimes_N \mathcal{K}_{g^{-1}})_N \cong {}_N\overline{\text{span}}(Nu_g \otimes u_{g^{-1}}N)_N \oplus \left(\bigoplus_{i=1}^{l(g)-1} {}_N(\mathcal{K}_{ga^ig^{-1}})_N \right). $$

Let us now show that $\overline{\text{span}}(Nu_g \otimes u_{g^{-1}}N)$ contains an N-N-subbimodule that is isomorphic with $\bigoplus_{i=0}^{r(g)-1} {}_N(\mathcal{K}_{\omega_g^i})_N$. For that, define $\xi_{g,i} \in \overline{\text{span}}(Nu_g \otimes u_{g^{-1}}N)$ as

$$ \xi_{g,i} := \frac{1}{\sqrt{r(g)}} \sum_{j=0}^{r(g)-1} (\omega_g^{-i})^j (u_{a^jg} \otimes u_{a^jg}^*). $$

Note that $x\xi_{g,i} = \xi_{g,i}\alpha_{\omega_g^i}(x)$ for every $x \in N$. Let $x, y \in N$ and $0 \leq i, j < r(g)$. We can make the following calculation:

$$ \begin{aligned} \langle \xi_{g,i}x, \xi_{g,j}y \rangle &= \sum_{k,l=0}^{r(g)-1} \frac{1}{r(g)} \omega_g^{-ki+lj} \langle u_{a^kg} \otimes u_{a^kg}^* x, u_{a^lg} \otimes u_{a^lg}^* y \rangle \\ &= \sum_{k,l=0}^{r(g)-1} \frac{1}{r(g)} \omega_g^{-ki+lj} \langle u_{a^kg} E_N(u_{a^kg}^* x y^* u_{a^lg}), u_{a^lg} \rangle \\ &= \sum_{k=0}^{r(g)-1} \frac{1}{r(g)} \omega_g^{-ki+kj} \langle u_{a^kg} E_N(u_{a^kg}^* x y^* u_{a^kg}), u_{a^kg} \rangle \\ &= \sum_{k=0}^{r(g)-1} \frac{1}{r(g)} \omega_g^{k(j-i)} \tau(xy^*). \end{aligned} \tag{3.7} $$

If $i \neq j$, then we get from (3.7) that $\overline{\xi_{g,i}N}$ is orthogonal to $\overline{\xi_{g,j}N}$. If $i = j$, then (3.7) implies that

$$ {}_N(\mathcal{K}_{\omega_g^i})_N \cong {}_N(\overline{\xi_{g,i}N})_N. $$

Hence, we indeed have that $\overline{\text{span}}(Nu_g \otimes u_{g^{-1}}N)$ contains an N-N-subbimodule isomorphic with $\bigoplus_{i=0}^{r(g)-1} {}_N(\mathcal{K}_{\omega_g^i})_N$. Putting everything together, we have found that ${}_N(\mathcal{K}_g \otimes_N \mathcal{K}_{g^{-1}})_N$ contains a subbimodule isomorphic with

$$ \left(\bigoplus_{i=0}^{r(g)-1} {}_N(\mathcal{K}_{\omega_g^i})_N \right) \oplus \left(\bigoplus_{i=1}^{l(g)-1} {}_N(\mathcal{K}_{ga^ig^{-1}})_N \right). $$

Since the right N-dimension of this subbimodule is the same as the right dimension of ${}_N(\mathcal{K}_g \otimes_N \mathcal{K}_{g^{-1}})_N$, namely $l(g)r(g)$, the two actually coincide.

We are left to prove that the bimodules in the decomposition are irreducible and pairwise nonisomorphic. The irreducibility follows immediately from Lemma 3.3.1. Let us now show that the subbimodules are all pairwise nonisomorphic. By Lemma 3.3.1 and the fact that the N-N-bimodules $\mathcal{K}_{\omega_g^i}$ are the only 1-dimensional bimodules in the decomposition, it suffices to show that $\mathcal{K}_{\omega_g^i}$ and $\mathcal{K}_{\omega_g^j}$ are nonisomorphic whenever $i \neq j$. For that, assume the existence of an N-N-bimodular isomorphism between $\mathcal{K}_{\omega_g^i}$ and $\mathcal{K}_{\omega_g^j}$. Then we see that there exists a unitary $u \in N$ such that $uxu^* = \alpha_{\omega_g^{i-j}}(x)$ for every $x \in N$. Note then that $u \in N'_{r(g)} \cap N = \mathbb{C}1$, and so $i = j$. This ends the proof. $\square$

We need one final result before we can start with the proof of the main theorem. For every $g \in \mathrm{BS}(n,m)$ we define $L(g)$ as the nonzero integer satisfying $ga^{L(g)}g^{-1} = a^{r(g)}$. Note that $L(g) \in \{l(g), -l(g)\}$.

Lemma 3.3.3. *Let $n, m \in \mathbb{Z}$ such that $2 \leq n \leq |m|$. Let (P, τ) be a tracial von Neumann algebra and let $\mathrm{BS}(n,m) \curvearrowright P$ be a trace preserving action such that $P \rtimes \langle a^{l(g)} \rangle \subset P \rtimes \mathrm{BS}(n,m)$ is irreducible for every $g \in \mathrm{BS}(n,m)$. Write $M := P \rtimes \mathrm{BS}(n,m)$, $N := P \rtimes \langle a \rangle$, $\Omega := \{e^{2\pi i s/r(g)} \mid s \in \mathbb{Z}, g \in \mathrm{BS}(n,m)\}$ and ${}_N(\mathcal{K}_g)_N := {}_N(\overline{\mathrm{span}}^{\|\cdot\|_2} N u_g N)_N$. Then for every $\omega, \mu \in \Omega$, we have that*

$$ {}_N(\mathcal{K}_\omega \otimes_N \mathcal{K}_g)_N \cong {}_N(\mathcal{K}_g \otimes_N \mathcal{K}_\mu)_N \text{ if and only if } \omega^{r(g)} = \mu^{L(g)}. $$

Proof. Fix $g \in \mathrm{BS}(n,m)$ and $\omega, \mu \in \Omega$. As before, we define $N_z := P \rtimes \langle a^z \rangle$ for every nonzero integer z.

We first prove the 'only if' part of the equivalence. So assume that ${}_N(\mathcal{K}_\omega \otimes_N \mathcal{K}_g)_N \cong {}_N(\mathcal{K}_g \otimes_N \mathcal{K}_\mu)_N$. For $0 \leq i < r(g)$, we denote by $\alpha_{i,\omega}$ the map $\mathrm{Ad}(u^*_{a^i g}) \circ \alpha_\omega$ on $N_{r(g)}$ and by $\alpha_{\mu,i}$ the map $\alpha_\mu \circ \mathrm{Ad}(u^*_{a^i g})$ on $N_{r(g)}$. We also define ${}_{N_{r(g)}}\mathcal{H}(\alpha_{i,\omega})_N$ by $\mathcal{H}(\alpha_{i,\omega}) = L^2(N)$ and $x\xi y = \alpha_{i,\omega}(x)\xi y$. In a similar way, we define ${}_{N_{r(g)}}\mathcal{H}(\alpha_{\mu,i})_N$. Note that

$$ {}_{N_{r(g)}}(\mathcal{K}_\omega \otimes_N \mathcal{K}_g)_N \cong \bigoplus_{i=0}^{r(g)-1} {}_{N_{r(g)}}\mathcal{H}(\alpha_{i,\omega})_N $$

and

$$ {}_{N_{r(g)}}(\mathcal{K}_g \otimes_N \mathcal{K}_\mu)_N \cong \bigoplus_{i=0}^{r(g)-1} {}_{N_{r(g)}}\mathcal{H}(\alpha_{\mu,i})_N. $$

Under these identifications, we find that set of all $N_{r(g)}$-N-bimodular elements of $\mathrm{B}(\mathcal{K}_\omega \otimes_N \mathcal{K}_g, \mathcal{K}_g \otimes_N \mathcal{K}_\mu)$ corresponds with

$$ B := \{[x_{i,j}]_{i,j} \mid x_{i,j} \in N \text{ and } x_{i,j}\alpha_{j,\omega}(x) = \alpha_{\mu,i}(x)x_{i,j} \text{ for all } x \in N_{r(g)}\}. $$

Take $r \in \{r(h) \mid h \in \mathrm{BS}(n,m)\}$ large enough such that $N_r \subset N_{r(g)}$, $\alpha_{i,\omega}(x) = \mathrm{Ad}(u^*_{a^i g})(x)$ and $\alpha_{\mu,i}(x) = \mathrm{Ad}(u^*_{a^i g})(x)$ for every $x \in N_r$. Then we see that $u_{a^i g} x_{i,j} u^*_{a^j g} \in N_r' \cap M = \mathbb{C}1$, whenever $[x_{i,j}]_{i,j} \in B$. Hence,

$$B = \{[x_{i,j}]_{i,j} \mid x_{i,j} \in \mathbb{C}\delta_{i,j} \text{ and } x_{i,i}\alpha_{i,\omega}(x) = \alpha_{\mu,i}(x)x_{i,i} \text{ for all } x \in N_{r(g)}\}.$$

Since ${}_N(\mathcal{K}_\omega \otimes_N \mathcal{K}_g)_N \cong {}_N(\mathcal{K}_g \otimes_N \mathcal{K}_\mu)_N$, there exists a unitary element $[x_{i,j}]_{i,j}$ inside B. For this unitary element, we have that $x_{0,0} \in \mathbb{T}1$ and $x_{0,0}\alpha_{0,\omega}(x) = \alpha_{\mu,0}(x)x_{0,0}$ for every $x \in N_{r(g)}$. From this we get that

$$\alpha_{0,\omega}(x) = \alpha_{\mu,0}(x), \text{ for all } x \in N_{r(g)}.$$

In particular, we have that $\alpha_{0,\omega}(u_{a^{r(g)}}) = \alpha_{\mu,0}(u_{a^{r(g)}})$. Now $\alpha_{0,\omega}(u_{a^{r(g)}}) = \omega^{r(g)} u_{a^{L(g)}}$, while $\alpha_{\mu,0}(u_{a^{r(g)}}) = \mu^{L(g)} u_{a^{L(g)}}$. Hence we have that $\omega^{r(g)} = \mu^{L(g)}$.

Let us now show the 'if' part of the equivalence. So assume that $\omega^{r(g)} = \mu^{L(g)}$. Using [FR12, Lemma 2.8], we can view $\mathcal{K}_g$ as an N-N-subbimodule of $L^2(N) \otimes_{N_{r(g)}} u_g L^2(N)$. Since both bimodules have the same right N-dimension, we see that $\mathcal{K}_g$ and $L^2(N) \otimes_{N_{r(g)}} u_g L^2(N)$ are actually isomorphic. Now define for every normal $*$-homomorphism $\alpha : N_{r(g)} \to N$ the bimodule ${}_{N_{r(g)}}\mathcal{H}(\alpha)_N$ by $\mathcal{H}(\alpha) = L^2(N)$ and $x\xi y = \alpha(x)\xi y$. Then

$${}_{N_{r(g)}}\mathcal{H}(\mathrm{Ad}(u_g^*))_N \cong {}_{N_{r(g)}} u_g L^2(N)_N.$$

Hence, $\mathcal{K}_g$ is isomorphic with $L^2(N) \otimes_{N_{r(g)}} \mathcal{H}(\mathrm{Ad}(u_g^*))$ as an N-N-bimodule. But then,

$${}_N(\mathcal{K}_\omega \otimes_N \mathcal{K}_g)_N \cong {}_N(L^2(N) \otimes_{N_{r(g)}} \mathcal{H}(\mathrm{Ad}(u_g^*) \circ \alpha_\omega))_N$$

and

$${}_N(\mathcal{K}_g \otimes_N \mathcal{K}_\mu)_N \cong {}_N(L^2(N) \otimes_{N_{r(g)}} \mathcal{H}(\alpha_\mu \circ \mathrm{Ad}(u_g^*)))_N.$$

Since we assumed that $\omega^{r(g)} = \mu^{L(g)}$, we have that $\mathrm{Ad}(u_g^*) \circ \alpha_\omega$ and $\alpha_\mu \circ \mathrm{Ad}(u_g^*)$ coincide on $N_{r(g)}$. Therefore ${}_{N_{r(g)}}\mathcal{H}(\mathrm{Ad}(u_g^*) \circ \alpha_\omega)_N \cong {}_{N_{r(g)}}\mathcal{H}(\alpha_\mu \circ \mathrm{Ad}(u_g^*))_N$. So ${}_N(\mathcal{K}_\omega \otimes_N \mathcal{K}_g)_N$ is isomorphic with ${}_N(\mathcal{K}_g \otimes_N \mathcal{K}_\mu)_N$. □

We finally present the proof of the main theorem.

Proof of Theorem C. For $i = 1,2$, write $\Gamma_i := \mathrm{BS}(n_i, m_i)$, $M_i := P_i \rtimes \Gamma_i$, $N_i := P_i \rtimes \langle a_i \rangle$ and $N_{i,z} := P_i \rtimes \langle a_i^z \rangle$ for every nonzero integer z. Interchanging if necessary the roles of M_1 and M_2, there exists a projection $p \in N_1$ and a $*$-isomorphism $\alpha : pM_1p \to M_2$. By Theorem 3.2.3, we may assume that $\alpha(pN_1p) = N_2$.

For $i \in \{1,2\}$ and $g \in \Gamma_i$, we define the N_i-N_i-bimodule $\mathcal{K}_g^i := \overline{\text{span}}^{\|\cdot\|_2} N_i u_g N_i$. By Lemma 3.3.1, we have that the sets $\{\mathcal{K}_g^2 \mid g \in \Gamma_2\}$ and $\{\alpha(p\mathcal{K}_g^1 p) \mid g \in \Gamma_1\}$ are the same. Looking at the left and right dimensions of the bimodules in both sets, we get that $\{(l(g), r(g)) \mid g \in \Gamma_1\} = \{(l(g), r(g)) \mid g \in \Gamma_2\}$.

Note that $n_i = \min(\{l(g) \mid g \in \Gamma_i\} \backslash \{1\})$ and so $n_1 = n_2$. On the other hand, $\{l(g)/r(g) \mid g \in \Gamma_i\} = (n_i/|m_i|)^{\mathbb{Z}}$. Therefore, also $\frac{n_1}{|m_1|} = \frac{n_2}{|m_2|}$. Together, this shows that $n_1 = n_2$ and $|m_1| = |m_2|$. It remains to prove that $m_1 = m_2$ whenever $n_1 \neq |m_1|$.

Whenever Q is a II_1 factor, we write $\text{Bimod}_f(Q)$ for the C^*-tensor category of all finite index Q-Q-bimodules (see e.g. [NT13] for the basics on C^*-tensor categories). If $\mathcal{H}$ is a nonzero Q-Q-bimodule, then we write $\text{Bimod}_{\mathcal{H}}(Q)$ for the C^*-tensor subcategory of $\text{Bimod}_f(Q)$ generated by all finite index Q-Q-subbimodules of $\mathcal{H}$.

Since $\alpha : pM_1p \to M_2$ satisfies $\alpha(pN_1p) = N_2$, we have that α gives rise to an equivalence between the C^*-tensor categories $\text{Bimod}_{L^2(pM_1p)}(pN_1p)$ and $\text{Bimod}_{L^2(M_2)}(N_2)$. On the other hand, the map $\mathcal{H} \to p\mathcal{H}p$ also gives an equivalence between $\text{Bimod}_{L^2(M_1)}(N_1)$ and $\text{Bimod}_{L^2(pM_1p)}(pN_1p)$. Altogether we have an equivalence β between the C^*-tensor categories $\text{Bimod}_{L^2(M_1)}(N_1)$ and $\text{Bimod}_{L^2(M_2)}(N_2)$.

We already know that β is a bijection between $\{\mathcal{K}_g^1 \mid g \in \Gamma_1\}$ and $\{\mathcal{K}_g^2 \mid g \in \Gamma_2\}$. Hence, we can choose a map $\sigma : \Gamma_1 \to \Gamma_2$ satisfying $\beta(\mathcal{K}_g^1) = \mathcal{K}_{\sigma(g)}^2$ for every $g \in \Gamma_1$. Note that since β preserves contragredients, we have for every $g \in \Gamma_1$ that $\mathcal{K}^2_{\sigma(g^{-1})} = \mathcal{K}^2_{\sigma(g)^{-1}}$. Also note that $r(g) = r(\sigma(g))$ and $l(g) = l(\sigma(g))$ for every $g \in \Gamma_1$, since β preserves left and right dimensions.

Write $\mathcal{F} := \{r(g) \mid g \in \Gamma_1\} \setminus \{1\} = \{r(g) \mid g \in \Gamma_2\} \setminus \{1\}$ and define the group Ω by

$$\Omega := \{\omega \in \mathbb{C} \mid \omega^f = 1 \text{ for some } f \in \mathcal{F}\}.$$

By Lemma 3.3.2, we have that the group of 1-dimensional subbimodules of $\{\mathcal{K}_g^1 \otimes_{N_1} \mathcal{K}_{g^{-1}}^1 \mid g \in \Gamma_1\}$ is exactly $\{\mathcal{K}_\omega^1 \mid \omega \in \Omega\}$. Similarly, the group of all 1-dimensional subbimodules of $\{\mathcal{K}_g^2 \otimes_{N_2} \mathcal{K}_{g^{-1}}^2 \mid g \in \Gamma_2\}$ is $\{\mathcal{K}_\omega^2 \mid \omega \in \Omega\}$. Since $\beta(\{\mathcal{K}_g^1 \otimes_{N_1} \mathcal{K}_{g^{-1}}^1 \mid g \in \Gamma_1\})$ coincides with $\{\mathcal{K}_g^2 \otimes_{N_2} \mathcal{K}_{g^{-1}}^2 \mid g \in \Gamma_2\}$, we have that $\beta(\{\mathcal{K}_\omega^1 \mid \omega \in \Omega\}) = \{\mathcal{K}_\omega^2 \mid \omega \in \Omega\}$. In this way, β gives rise to an automorphism $\Delta : \Omega \to \Omega$.

Now define for $g \in \Gamma_1$ and $h \in \Gamma_2$ the sets

$$W_g^1 := \{(\omega, \mu) \in \Omega \times \Omega \mid \mathcal{K}_\omega^1 \otimes_{N_1} \mathcal{K}_g^1 \cong \mathcal{K}_g^1 \otimes_{N_1} \mathcal{K}_\mu^1\}$$

and

$$W_h^2 := \{(\omega, \mu) \in \Omega \times \Omega \mid \mathcal{K}_\omega^2 \otimes_{N_2} \mathcal{K}_h^2 \cong \mathcal{K}_h^2 \otimes_{N_2} \mathcal{K}_\mu^2\}.$$

We have that $(\Delta \times \Delta)(W^1_g) = W^2_{\sigma(g)}$ for every $g \in \Gamma_1$. Using Lemma 3.3.3 this implies that

$$(\Delta \times \Delta)(\{(\omega, \mu) \in \Omega \times \Omega \mid \omega^{r(g)} = \mu^{L(g)}\})$$
$$= \{(\omega, \mu) \in \Omega \times \Omega \mid \omega^{r(\sigma(g))} = \mu^{L(\sigma(g))}\}. \tag{3.8}$$

Now assume, by way of reaching a contradiction, that $n_1 = n_2$ and $m_1 = -m_2$ with $n_1 \neq m_1$. Put $n := n_1$, $m := m_1$, $k := \gcd(n, |m|)$, $n_0 := n/k$ and $m_0 := m/k$. By taking $g \in \Gamma_1$ equal to b^{-1} in equation (3.8), we see that

$$\Delta \times \Delta(\{(\omega, \mu) \in \Omega \times \Omega \mid \omega^n = \mu^m\})$$
$$= \{(\omega, \mu) \in \Omega \times \Omega \mid \omega^{r(\sigma(b^{-1}))} = \mu^{L(\sigma(b^{-1}))}\}.$$

We already know that $r(\sigma(g)) = r(g)$ and $l(\sigma(g)) = l(g)$ for every $g \in \Gamma_1$. Therefore $r(\sigma(b^{-1})) = n$ and $L(\sigma(b^{-1})) \in \{m, -m\}$. Since $r(h)/L(h) \in (-n/m)^{\mathbb{Z}}$ for every $h \in \Gamma_2$, we get that $L(\sigma(b^{-1}))$ must be equal to $-m$.

Take t such that $|n_0^t m_0^t| > 2$. Define $\omega := e^{2\pi i/(kn_0^{t+1}m_0^t)}$ and $\mu := e^{2\pi i/(kn_0^t m_0^{t+1})}$. Then $\omega^n = \mu^m$ and hence

$$\Delta(\mu)^{-m} = \Delta(\omega)^n = \Delta(\omega^n) = \Delta(\mu^m) = \Delta(\mu)^m.$$

Therefore $\Delta(\mu)^{2m} = 1$, or equivalently $\mu^{2m} = 1$. This is a contradiction, since $|n_0^t m_0^t| > 2$. We conclude that also $m_1 = m_2$ whenever $n_1 \neq m_1$. □

3.4 Two comments on the assumptions of the main theorem

In this section, we examine the assumptions on $\mathrm{BS}(n, m) \curvearrowright P$ found in the main theorem. We show that whenever P is abelian, these are equivalent to some seemingly weaker/stronger assumptions.

Throughout this section, let n and m be integers such that $2 \leq n \leq |m|$. Let k be the greatest common divisor of n and $|m|$. Write $n_0 = n/k$, $m_0 = m/k$ and $\mathcal{F} = \{kn_0^s|m_0|^t \mid s, t \in \mathbb{N}, s + t > 0\} = \{l(g) \mid g \in \mathrm{BS}(n, m)\} \setminus \{1\}$.

Recall from Lemma 1.11.5 that the quasi-centralizer of $\langle a\rangle$ in $\mathrm{BS}(n, m)$ is given by $\mathrm{QC}_{\mathrm{BS}(n,m)}(\langle a\rangle) = \{g \in \Gamma \mid ga^{l(g)}g^{-1} = a^{l(g)}\}$. We have the following result.

Lemma 3.4.1. *Let* $\mathrm{BS}(n, m) \curvearrowright (X, \mu)$ *be a pmp action of* $\mathrm{BS}(n, m)$ *on a standard probability space* X. *Write* $\Gamma := \mathrm{BS}(n, m)$, $\Lambda := \mathrm{QC}_\Gamma(\langle a\rangle)$, $M := L^\infty(X) \rtimes \Gamma$ *and* $N_z := L^\infty(X) \rtimes \langle a^z\rangle$ *for every* $z \in \mathbb{Z} \setminus \{0\}$. *The following statements are equivalent.*

1. *$N_z' \cap M = \mathbb{C}1$ for every $z \in \mathcal{F}$.*
2. *$\Lambda \curvearrowright X$ is essentially free and $\langle a^z \rangle \curvearrowright X$ is ergodic for every $z \in \mathcal{F}$.*
3. *$\Gamma \curvearrowright X$ is essentially free and $\langle a^z \rangle \curvearrowright X$ is ergodic for every $z \in \mathcal{F}$.*

Proof. $1 \Rightarrow 2$. Note that every $\langle a^z \rangle$-invariant element of $L^\infty(X)$ is an element of $N_z' \cap M = \mathbb{C}1$. Therefore $\langle a^z \rangle \curvearrowright X$ is ergodic for every $z \in \mathcal{F}$. It remains to prove that $\Lambda \curvearrowright X$ is essentially free. For every $g \in \mathrm{BS}(n,m)$, we write $\mathrm{Fix}(g)$ for the fixed point set of g, i.e. $\mathrm{Fix}(g) := \{x \in X \mid x = g \cdot x\}$. Assume, by way of reaching a contradiction, that $\Lambda \curvearrowright X$ is not essentially free. Then there exists an element $g \in \Lambda \setminus \{e\}$ such that $\mu(\mathrm{Fix}(g)) > 0$. Since $\mathrm{Fix}(g)$ is an $\langle a^{l(g)} \rangle$-invariant Borel subset of X, we get that $\mu(\mathrm{Fix}(g)) = 1$. To reach a contradiction, observe that u_g is a nontrivial element of $N_{l(g)}' \cap M$.

$2 \Rightarrow 3$. If $\Gamma = \Lambda$, there is clearly nothing to prove. So assume that $\Gamma \neq \Lambda$. Let $g \in \Gamma \setminus \Lambda$ and assume, by way of reaching a contradiction, that $\mu(\mathrm{Fix}(g)) > 0$. Take a nonzero integer z such that $\mu(a^z \cdot \mathrm{Fix}(g) \ \cap \ \mathrm{Fix}(g)) > 0$. Note that $a^z \cdot \mathrm{Fix}(g) \ \cap \ \mathrm{Fix}(g) \subset \mathrm{Fix}(ga^zg^{-1}a^{-z})$. Therefore $\mu(\mathrm{Fix}(ga^zg^{-1}a^{-z})) > 0$. On the other hand $ga^zg^{-1}a^{-z}$ belongs to Λ, since Λ is a normal subgroup of Γ. Furthermore $ga^zg^{-1}a^{-z}$ is nontrivial, since g would otherwise belong to $\mathcal{C}_\Gamma(\langle a^z \rangle) \subset \Lambda$. Altogether we have reached a contradiction.

$3 \Rightarrow 1$. Since $\Gamma \curvearrowright X$ is essentially free, we have that $L^\infty(X)' \cap M = L^\infty(X)$. Therefore $N_z' \cap M \subset L^\infty(X)$ for every $z \in \mathcal{F}$. But then, for every $z \in \mathcal{F}$, we have that $N_z' \cap M$ is the von Neumann algebra of $\langle a^z \rangle$-invariant functions of $L^\infty(X)$. The ergodicity of $\langle a^z \rangle \curvearrowright X$ now finishes the proof. □

We also have the following result.

Lemma 3.4.2. *Let $\mathrm{BS}(n,m) \curvearrowright (X,\mu)$ be a pmp action of $\mathrm{BS}(n,m)$ on a standard probability space X. If $\langle a^k \rangle \curvearrowright X$ is ergodic, then $\langle a^z \rangle \curvearrowright X$ is ergodic for every $z \in \mathcal{F}$.*

Proof. Assume that $\langle a^k \rangle \curvearrowright X$ is ergodic. For every $z \in \mathbb{Z} \setminus \{0\}$, we denote by $P(z)$ the $\langle a^z \rangle$-invariant elements of $L^\infty(X)$. By assumption we have that $P(k) = \mathbb{C}1$. To prove the lemma, we need to show that $P(kn_0^s m_0^t) = \mathbb{C}1$ for every $s,t \in \mathbb{N}$ with $s+t > 0$. So fix $s,t \in \mathbb{N}$ with $s+t > 0$ and note that

$$P(kn_0^s m_0^t) = u_{b^s}^* P(km_0^{t+s}) u_{b^s} \text{ and } P(kn_0^s m_0^t) = u_{b^t} P(kn_0^{s+t}) u_{b^t}^*.$$

In particular, we have that

$$\dim(P(km_0^{t+s})) = \dim(P(kn_0^s m_0^t)) = \dim(P(kn_0^{s+t})).$$

Since n_0 and m_0 are coprime, it suffices to show that $\dim(P(kz))$ divides z whenever $z \in \mathbb{Z} \setminus \{0\}$. To that end, fix $z \in \mathbb{Z} \setminus \{0\}$ and note that the action $(\langle a^k \rangle / \langle a^{kz} \rangle) \curvearrowright P(kz)$ is ergodic since $P(k) = \mathbb{C}1$. Write $L^\infty(Y, \mu)$ for $P(kz)$ and let $(\langle a^k \rangle / \langle a^{kz} \rangle) \curvearrowright Y$ be the ergodic action corresponding to $(\langle a^k \rangle / \langle a^{kz} \rangle) \curvearrowright P(kz)$. Then Y is purely atomic. Indeed, if not, then Y would contain a Borel subset Z with $0 < \mu(Z) < |z|$. This in turn would mean that

$$\begin{aligned} 1 = \mu(Y) &= \mu((\langle a^k \rangle / \langle a^{kz} \rangle) \cdot Z) \\ &\leq z\, \mu(Z) < 1. \end{aligned}$$

Let $y \in Y$ be an atom. Then by the orbit-stabilizer theorem, we have that the number of elements in the orbit of y is a divisor of $|\langle a^k \rangle / \langle a^{kz} \rangle| = z$. Since the action is ergodic, the orbit of y is the whole of Y. Hence we find that Y consists of exactly j atoms, where j is some divisor of z. In other words, $P(kz)$ must be finite dimensional and its dimension should divide z. This ends the proof. $\square$

3.5 Crossed products with at least two Cartan subalgebras

As promised we give some examples of essentially free ergodic pmp actions of Baumslag-Solitar groups where $\langle a \rangle \curvearrowright X$ does not act ergodically and such that the corresponding crossed product has at least two Cartan subalgebras up to unitary conjugacy.

Let Γ be a countable discrete group with an infinite abelian almost normal subgroup Λ such that $\bigcap_{g \in \Gamma} g \Lambda g^{-1}$ is trivial, e.g. $\Gamma = \mathrm{BS}(n, m)$ and $\Lambda = \langle a \rangle$ when $|n| \neq |m|$. The following result was communicated to us by Anna Krogager.

Theorem 3.5.1. *Let Γ be a countable discrete group with an infinite abelian almost normal subgroup Λ such that $\bigcap_{g \in \Gamma} g \Lambda g^{-1}$ is trivial. Then there exists an essentially free ergodic pmp action of Γ on some standard probability space X such that the crossed product $L^\infty(X) \rtimes \Gamma$ has at least two Cartan subalgebras that are not unitarily conjugate.*

Let us give the construction of $\Gamma \curvearrowright X$. We start by recalling the concept of a co-induced action. Assume that Λ acts on a probability space (X, μ). Choose a section $\theta : \Lambda \backslash \Gamma \to \Gamma$ such that $\theta(\Lambda) = e$ and let $r : \Gamma \to \Lambda$ be the unique map satisfying $g = r(g)\theta(\Lambda g)$. We define a cocycle $\Omega : \Lambda \backslash \Gamma \times \Gamma \to \Lambda$ by $\Omega(\Lambda t, g) = r(t)^{-1} r(tg)$. The formula

$$(g \cdot x)_{\Lambda t} = \Omega(\Lambda t, g) \cdot x_{\Lambda t g}, \quad x \in X^{\Lambda \backslash \Gamma}$$

then gives a well-defined action of Γ on $X^{\Lambda\backslash\Gamma}$ called the *co-induced action* of $\Lambda \curvearrowright X$ to Γ.

Next, we recall the construction of the relative profinite completion of Γ with respect to Λ introduced in [Sc80]. Denote by $\mathrm{Sym}(\Lambda\backslash\Gamma)$ the group of all permutations of the countable set $\Lambda\backslash\Gamma$ endowed with the topology of pointwise convergence. We have a homomorphism $\pi : \Gamma \to \mathrm{Sym}(\Lambda\backslash\Gamma)$ induced by the right multiplication. Note that $\ker(\pi) = \bigcap_{g\in\Gamma} g\Lambda g^{-1}$ is trivial and so π is injective. Let G denote the closure of $\pi(\Gamma)$ inside $\mathrm{Sym}(\Lambda\backslash\Gamma)$ and let K denote the closure of $\pi(\Lambda)$. Then G is a locally compact group called the *relative profinite completion* of Γ with respect to Λ. Furthermore, K an abelian compact open almost normal subgroup of G. Since π was assumed to be injective, we may regard Γ as a dense subgroup of G and Λ as a dense subgroup of K.

We consider the action of Γ on $X := K^{\Lambda\backslash\Gamma}$ co-induced from the translation action $\Lambda \curvearrowright K$. This is an essentially free ergodic pmp action such that $\Lambda \curvearrowright X$ is not ergodic. Put $M := L^\infty(X) \rtimes \Gamma$. In forthcoming work of Anna Krogager, it is shown that the subalgebra generated by $L(\Lambda)$ and the Λ-invariant functions $L^\infty(X)^\Lambda$ is a Cartan subalgebra of M that is not unitarily conjugate with $L^\infty(X)$.

Chapter 4

Conclusion

In this thesis we examined both the group von Neumann algebras of the Baumslag-Solitar groups and the crossed product von Neumann algebras of some of their actions. In Chapter 2, we showed that the rational number $|n/m|$ is an invariant of $L(\mathrm{BS}(n,m))$. Concretely, if $L(\mathrm{BS}(n,m))$ and $L(\mathrm{BS}(p,q))$ are isomorphic and nonamenable, then $|n/m| = |p/q|^{\pm 1}$. In Chapter 3, we gave a rigidity result for crossed products arising from certain actions of Baumslag-Solitar groups. For $i \in \{1,2\}$, let $2 \leq n_i \leq |m_i|$ and let $\mathrm{BS}(n_i,m_i) \curvearrowright (X_i,\mu_i)$ be an essentially free ergodic pmp action such that $\langle a_i \rangle \curvearrowright X_i$ is aperiodic. If the crossed products $L^\infty(X_1) \rtimes \mathrm{BS}(n_1,m_1)$ and $L^\infty(X_2) \rtimes \mathrm{BS}(n_2,m_2)$ are isomorphic, then

- $n_1 = n_2$ and $m_1 = m_2$, if $n_1 \neq |m_1|$;
- $n_1 = |m_1| = n_2 = |m_2|$, if $n_1 = |m_1|$.

There is however more that can be done. It turns out that the Baumslag-Solitar groups are part of a larger class of groups called the *generalized Baumslag-Solitar groups*. A generalized Baumslag-Solitar group (or *GBS group*) is a group that acts on a tree with infinite cyclic edge and vertex stabilizers. The action of the GBS group on the tree can entirely be described by a labelled graph. This in turn corresponds with a presentation of the group, that in some sense looks like a combination of HNN extensions and amalgamated free products of infinite cyclic groups. We refer to the preliminary section of [CF08] for the most basic facts on GBS groups.

As we did with the Baumslag-Solitar groups, one could also try to classify both the group von Neumann algebras of the GBS groups and the crossed product von Neumann algebras of some of their actions. The difficulty here will lie in the fact that it is hard to determine whether two given labelled graphs define isomorphic groups. So even on the group level it is not easy to give a classification. However one might hope to retrieve certain group invariants from the von Neumann algebras.

Bibliography

[An95] C. Anantharaman-Delaroche, Amenable correspondences and approximation properties for von Neumann algebras. *Pacific J. Math.* **171** (1995), 309–341.

[Bo07] V.I. Bogachev, Measure theory. Vol. II. Springer-Verlag, 2007.

[BO08] N.P. Brown and N. Ozawa, C^*-algebras and finite-dimensional approximations. *Graduate Studies in Mathematics* **88**, American Mathematical Society, 2008.

[BS62] G. Baumslag and D. Solitar, Some two-generator one-relator non-Hopfian groups. *Bull. Amer. Math. Soc.* **68** (1962) 199-201.

[BV12] M. Berbec and S. Vaes, W^*-superrigidity for group von Neumann algebras of left-right wreath products. *Proc. Lond. Math. Soc.* **108** (2014), 1116-1152.

[Br63] J.L. Britton, The word problem. *Ann. Math.* **77** (1963), 16-32.

[CH08] I. Chifan and C. Houdayer, Bass-Serre rigidity results in von Neumann algebras. *Duke Math. J.* **153** (2010), 23-54.

[CF08] M. Clay and M. Forester, On the isomorphism problem for generalized Baumslag-Solitar groups. *Algebr. Geom. Topol.* **8** (2008), 2289-2322

[Co75] A. Connes, Classification of injective factors. *Ann. Math.* **104** (1976), 73-115.

[Co80] A. Connes, Classification des facteurs. In *Operator algebras and applications, Part 2 (Kingston, 1980)*, Proc. Sympos. Pure Math. **38**, Amer. Math. Soc., Providence, 1982, pp. 43-109.

[Co94] A. Connes, Noncommutative geometry, Academic Press, 1994.

[CFW81] A. Connes, J. Feldman and B. Weiss, An amenable equivalence relation is generated by a single transformation. *Ergodic Theory Dynam. Systems* **4** (1981), 431-450.

[CH88] M. Cowling and U. Haagerup, Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. *Invent. Math.* **96** (1989), 507-549.

[DS09] A. Danilenko and C.E. Silva, Ergodic theory: nonsingular transformations. *Encyclopedia of Complexity and Systems Science*, Springer, New York, (2009), 3055–3083.

[dlHP11] P. De La Harpe and J.-P. Préaux, C*-simple groups: amalgamated free products, HNN extensions, and fundamental groups of 3-manifolds. *J. Topol. Anal.* **3** (2011), 451-489.

[Di54] J. Dixmier, Sous-Anneaux Abéliens Maximaux Dans Les Facteurs De Type Fini. *Ann. Math.* **59** (1954), 279-286.

[Di81] J. Dixmier, Von Neumann Algebras. *North-Holland Mathematical Library* **27**, North-Holland Publishing Co., 1981.

[Dy94] K. Dykema, Interpolated free group factors. *Pac. J. Math.* **163** (1994), 123-135.

[Dy95] K. Dykema, Crossed product decompositions of a purely infinite von Neumann algebra with faithful, almost periodic weight. *Indiana University Mathematics Journal* **44** (1995), 433-450.

[ECH+92] D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson and W. Thurston, Word Processing in Groups. *AK Peters, Ltd.*, 1992.

[Fa09] S. Falguières, Outer automorphism groups and bimodule categoriesof type II_1 factors. *PhD thesis, Katholieke Universiteit Leuven*, `www.math.unicaen.fr/~falguieres/docs/MaThese.pdf`

[FR12] S. Falguières and S. Raum, Tensor-categories arising as bimodule categories of II_1 factors. *Adv. Math.* **237** (2013), 331-359.

[FM98] B. Farb and L. Mosher, A rigidity theorem for the solvable Baumslag-Solitar groups (With an appendix by Daryl Cooper). *Invent. Math.* **131** (1998), 419-451.

[FM77] J. Feldman and C.C. Moore, Ergodic equivalence relations, cohomology, and von Neumann algebras. *Trans. Amer. Math. Soc* **234** (1977), 289-324.

[FM77b] J. Feldman and C.C. Moore, Ergodic equivalence relations, cohomology, and von Neumann algebras II. *Trans. Amer. Math. Soc.* **234** (1977), 325-359.

[FSZ89] J. Feldman, C. Sutherland and R. Zimmer, Subrelations of ergodic equivalence relations. *Ergodic Theory Dynam. Systems* **9** (1989), 239-269.

[FV12] P. Fima and S. Vaes, HNN extensions and unique group measure space decomposition of II_1 factors. *Trans. Amer. Math. Soc.* **364** (2012), 2601-2617.

[Ga99] D. Gaboriau, Coût des relations d'équivalence et des groupes. *Invent. Math.* **139** (2000), 41-98.

[Ga01] D. Gaboriau, Invariants l^2 de relations d'équivalence et de groupes. *Publ. Math. Inst. Hautes Études Sci.* **95** (2002), 93-150.

[GS91] S.M. Gersten and HB Short, Rational subgroups of biautomatic groups. *Ann. Math.* **134** (1991), 125-158.

[Gr93] M. Gromov, Asymptotic invariants of infinite groups. *Geometric group theory* **182**, London Math. Soc. Lecture Note Ser., 1993.

[HK88] T. Hamachi and H. Kosaki, Index and flow of weights of factors of type III. *Proc. Japan Academy* **64** (1988), 11-13.

[HNN49] G. Higman, B.H. Neumann and H. Neumann, Embedding theorems for groups. *London Math. Soc.* **4** (1949), 247-254.

[Ho13] C. Houdayer, An introduction to von Neumann algebras. *Lecture Notes Summer school on rigidity and group actions, Paris, 2013*, `cyrilhoudayer.files.wordpress.com/2014/09/paris-2013.pdf`

[HR13] C. Houdayer and S. Raum, Baumslag-Solitar groups, relative profinite completions and measure equivalence rigidity. *J. Topol.* **8** (2015), 295-313.

[Io12] A. Ioana, Cartan subalgebras of amalgamated free product II_1 factors. *Ann. Sci. Éc. Norm. Supér.* **48** (2015), 71-130.

[IPP08] A. Ioana, J. Peterson and S. Popa, Amalgamated free products of weakly rigid factors and calculation of their symmetry groups. *Acta Math.* **200** (2008), 85-153.

[IPV10] A. Ioana, S. Popa and S. Vaes, A class of superrigid group von Neumann algebras. *Ann. Math.* **178** (2013), 231-286.

[Jo83] V.F.R. Jones, Index for subfactors. *Invent. Math.* **72** (1983), 1-25.

[JS97] V. Jones and V.S. Sunder, Introduction to subfactors. *London Mathematical Society Lecture Note Series* **234**, Cambridge University Press, Cambridge, 1997.

[Ke95] A.S. Kechris, Classical descriptive set theory. *Graduate Texts in Mathematics* **156**, Springer-Verlag, New York, 1995.

[Ki11] Y. Kida, Invariants of orbit equivalence relations and Baumslag-Solitar groups. *Tohoku Math. J.* **66** (2014), 205-258.

[Ma63] D. Maharam, Incompressible transformations. *Fund. Math.* **56** (1964), 35-50.

[MV13] N. Meesschaert and S. Vaes, Partial classification of the Baumslag-Solitar group von Neumann algebras. *Doc. Math.* **19** (2014), 629-645.

[Mo91] D.I. Moldavanskii, Isomorphism of the Baumslag-Solitar groups. *Ukrainian Math. J.* **43** (1991), 1569-1571.

[MvN36] F. J. Murray and J. Von Neumann, On rings of operators, *Ann. Math.* **37** (1936), 116–229.

[MvN43] F. J. Murray and J. Von Neumann, On rings of operators IV, *Ann. Math.* **44** (1943), 716-808.

[NT13] S. Neshveyev and L. Tuset, Compact quantum groups and their representation categories. *Cours Spécialisés* **20** Société Mathématique de France, Paris, 2013

[OW80] D.S. Ornstein and B. Weiss, Ergodic theory of amenable group actions. *Bull. Amer. Math. Soc. (N.S)* **2** (1980), 161-164.

[OP07] N. Ozawa and S. Popa, On a class of II_1 factors with at most one Cartan subalgebra. *Ann. Math.* **72** (2010), 101-13.

[Po86] S. Popa, Correspondences, INCREST preprint No. 56/1986, `www.math.ucla.edu/~popa/preprints.html`

[Po93] S. Popa, Markov traces on universal Jones algebras and subfactors of finite index. *Invent. Math.* **111** (1993), 375-405.

[Po03] S. Popa, Strong rigidity of II_1 factors arising from malleable actions of $w-$rigid groups, I. *Invent. Math.* **165** (2006), 369-408.

[Po04] S. Popa, Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups, II. *Invent. Math.* **165** (2006), 409-452.

[PV11] S. Popa and S. Vaes, Unique Cartan decomposition for II_1 factors arising from arbitrary actions of free groups. *Acta Math.* **212** (2014), 141-198.

[Ra94] F. Rădulescu, Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group, of noninteger index. *Invent. Math.* **115** (1994), 347-389.

[Sc80] G. Schlichting, Operationen mit periodischen Stabilisatoren. *Arch. Math.* **34** (180), 97-99.

[Se80] J.-P. Serre, Trees. *Berlin-Heidelberg-New York: Springer* (1980).

[Si55] I.M. Singer, Automorphisms of finite factors. *Amer. J. Math.* **77** (1955), 117-133.

[St05] Y. Stalder, Moyennabilité intérieure et extensions HNN. *Ann. Inst. Fourier* **56** (2006), 309-323.

[Tak79] M. Takesaki, *Theory of operator algebras. I*, volume 124 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 1979. Operator Algebras and Non-commutative Geometry.

[Tak03] M. Takesaki, *Theory of operator algebras. II*, volume 125 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2003. Operator Algebras and Non-commutative Geometry.

[Tz03] K. Tzanev, Hecke C*-algebras and amenability. *J. Operator Theory* **50** (2003), 169-178.

[Ue04] Y. Ueda, HNN extensions of von Neumann algebras. *J. Funct. Anal.* **225** (2005), 383-426.

[Ue07] Y. Ueda, Remarks on HNN extensions in operator algebras. *Illinois J. Math.* **52** (2008), 705-725.

[Va08] S. Vaes, Explicit computations of all finite index bimodules for a family of II_1 factors. *Ann. Sci. Éc. Norm. Supér.* **41** (2008), 743-788.

[Va10] S. Vaes, One-cohomology and the uniqueness of the group measure space decomposition of a II_1 factor. *Math. Annal.* **355** (2013), 661-696.

[Va13] S. Vaes, Normalizers inside amalgamated free product von Neumann algebras. *Publ. Res. Inst. Math. Sci.* **50** (2014), 695–721.

[VDN92] D.V. Voiculescu, K.J. Dykema and A. Nica, Free random variables. CRM Monograph Series 1, American Mathematical Society, Providence, RI, 1992.

[vN29] J. von Neumann, Zur Algebra der Funktionaloperationen und Theorie der normalen Operatoren. *Math. Ann.* **102** (1930) 370-417.

[Wa82] P. Walters, An Introduction to Ergodic Theory. Springer, 1982.

[Wh01] K. Whyte, The large scale geometry of the higher Baumslag-Solitar groups. *Geom. Funct. Anal.* **11** (2001), 1327-1343.

[Zi84] R.J. Zimmer, Ergodic Theory and Semisimple Groups, Birkhäuser-Verlag, Boston, 1984.

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